Relations Between the Rectangular Components of Stress and Equality of Complementary Shear Stresses

When a body is subjected to external forces, internal forces develop across its imaginary internal surfaces to maintain equilibrium. The intensity of the internal force per unit area is called stress.

At any point in a three-dimensional elastic body, the state of stress is represented by nine rectangular components, which are classified into:

- Three normal stresses: $\sigma_x, \sigma_y, \sigma_z$.
- Six shear stresses: $\tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{zx}, \tau_{xz}$.

Here, the first subscript indicates the direction of the normal to the plane on which the stress acts, while the second subscript indicates the direction of the stress component.

We shall establish that the complementary shear stresses are equal, namely,

$$\boxed{\tau_{xy} = \tau_{yx}, \qquad \tau_{yz} = \tau_{zy}, \qquad \tau_{zx} = \tau_{xz}.}$$

Let us consider an infinitesimal rectangular parallelepiped of a material body having dimensions $dx,\ dy,\ dz$ along the $x,\ y,\ z$ axes, respectively.

The element is subjected to the following stress components:

- Normal stresses: $\sigma_x, \sigma_y, \sigma_z$.
- Shear stresses: $\tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{zx}, \tau_{xz}$.

The stress components are assumed to be continuously distributed over the faces of the element.

We first consider the equilibrium of moments about the z-axis.

(a) Moments about the (z)-axis

The shear stress τ_{xy} acts on the faces perpendicular to the x-axis in the y-direction.

The resultant force due to this stress on each face is

$$F_1 = \tau_{xy}(dy, dz).$$

The two forces acting on the opposite faces form a couple. The perpendicular distance between their lines of action is (dx).

Hence, the moment of this couple about the (z)-axis is

$$M_1 = \tau_{xy}(dy, dz), dx.$$

Similarly, the shear stress τ_{yx} acts on the faces perpendicular to the (y)-axis in the (x)-direction.

The resultant force is

$$F_2 = \tau_{yx}(dx, dz).$$

The corresponding couple has a perpendicular distance (dy). Its moment about the (z)-axis is

$$M_2 = \tau_{yx}(dx, dz), dy.$$

For rotational equilibrium about the (z)-axis,

$$\sum M_z = 0.$$

Taking the two couples in opposite senses,

$$\tau_{yx}(dx, dy, dz) = 0.$$

Therefore,

$$(\tau_{xy} - \tau_{yx})dx, dy, dz = 0.$$

Since the volume of the element is non-zero,

$$\boxed{\tau_{xy} = \tau_{yx}.}$$

Thus, the complementary shear stresses acting on the mutually perpendicular (x)- and (y)-planes are equal.

(b) Moments about the (x)-axis

Following the same procedure, consider the shear stresses τ_{yz} and τ_{zy}.

The moment due to τ_{yz} is

$$M_1 = \tau_{yz}(dz, dx), dy.$$

The moment due to τ_{zy} is

$$M_2 = \tau_{zy}(dy, dx), dz.$$

Applying rotational equilibrium about the (x)-axis,

$$\sum M_x = 0,$$

we obtain

$$\tau_{zy}(dx, dy, dz) = 0.$$

Consequently,

$$\boxed{\tau_{yz} = \tau_{zy}.}$$

(c) Moments about the (y)-axis

Similarly, consider the shear stresses τ_{zx} and τ_{xz}.

The moment due to τ_{zx} is

$$M_1 = \tau_{zx}(dx, dy), dz.$$

The moment due to τ_{xz} is

$$M_2 = \tau_{xz}(dz, dy), dx.$$

Applying rotational equilibrium about the (y)-axis,

$$\sum M_y = 0,$$

we get

$$\tau_{xz}(dx, dy, dz) = 0.$$

Hence,

$$\boxed{\tau_{zx} = \tau_{xz}.}$$

Combining the three results, we obtain

$$\boxed{\tau_{xy} = \tau_{yx},\ \tau_{yz} \quad = \tau_{zy},\ \tau_{zx} = \tau_{xz}.}$$

Therefore, the six rectangular shear stress components reduce to only three independent shear stress components.

Stress Tensor Representation

The general three-dimensional stress tensor is initially represented by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}.$$

Using the equality of complementary shear stresses, it becomes

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}.$$

Thus,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T.$$

The stress tensor is therefore symmetric, and a general three-dimensional stress state has only six independent components instead of nine.

Physical Significance

The equality of complementary shear stresses is a direct consequence of the conservation of angular momentum. If the complementary shear stresses were unequal, they would produce an unbalanced couple, resulting in angular acceleration of the infinitesimal element.

Hence, for a classical continuum in which no independent body couples or couple stresses are present, the complementary shear stresses must be equal to ensure rotational equilibrium.

Important conclusion: The equilibrium of moments about the three coordinate axes establishes the symmetry of the stress tensor and reduces the number of independent rectangular stress components from nine to six.