Rectangular Components of Stress

Stress is defined as the internal resisting force acting per unit area across an imaginary section of a body. At any point inside a body, the stress acting on an infinitesimal element can be resolved into six rectangular components, consisting of three normal stresses and three independent shear stresses.

1. Normal Stress

The stress component acting perpendicular (normal) to a particular plane is called normal stress and is denoted by σ. There are three normal stress components:

- σ_x: Normal stress acting perpendicular to the YZ-plane.
- σ_y: Normal stress acting perpendicular to the ZX-plane.
- σ_z: Normal stress acting perpendicular to the XY-plane.

Normal stress may be of two types: tensile stress, which tends to stretch or elongate the body, and compressive stress, which tends to shorten or compress the body.

2. Shear Stress

The stress component acting tangentially (parallel) to a particular plane is called shear stress and is denoted by τ. The three independent shear stress components are:

- τ_{xy}: Shear stress acting in the y-direction on the plane normal to the x-axis.
- τ_{yz}: Shear stress acting in the z-direction on the plane normal to the y-axis.
- τ_{zx}: Shear stress acting in the x-direction on the plane normal to the z-axis.

Shear stress tends to change the shape of a body by producing angular distortion without necessarily changing its volume.

3. Mathematical Representation

The complete state of stress at a point can be represented by the following stress tensor:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

For rotational equilibrium of an infinitesimal element, the complementary shear stresses are equal:

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{yz} = \tau_{zy}, \qquad \tau_{zx} = \tau_{xz}.$$

Thus, the stress tensor is symmetric and has six independent components.

4. Physical Significance of Rectangular Components of Stress

Component	**Physical Significance**
σ_x	Produces extension or compression along the x-direction.
σ_y	Produces extension or compression along the y-direction.
σ_z	Produces extension or compression along the z-direction.
τ_{xy}	Produces angular distortion in the xy-plane.
τ_{yz}	Produces angular distortion in the yz-plane.
τ_{zx}	Produces angular distortion in the zx-plane.

The six independent rectangular components of stress completely describe the state of stress at a point in a three-dimensional body. Normal stresses are responsible for stretching or compression, whereas shear stresses are responsible for distortion or change in shape.

RATE OF STRAIN TENSOR AND PRINCIPAL RATES OF STRAIN

1. Introduction to Rate of Strain

In continuum mechanics, when a body or fluid moves, the relative positions of its material particles may change with time. This change may involve an increase or decrease in the length of material lines or a change in the angle between two intersecting material lines. The rate of strain describes the rate at which these deformations take place with respect to time. It measures the rate of stretching, compression and angular distortion of a material element during motion.

The rate of strain tensor is also known as the **deformation rate tensor** or **stretching tensor**.

2. Velocity Field and Velocity Gradient

Let the velocity field of a fluid or deformable body be represented by

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$

where u, v and w are the velocity components along the x, y and z directions, respectively.

The velocity gradient tensor is given by

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}.$$

The velocity gradient tensor can be decomposed into a symmetric part and an antisymmetric part. The symmetric part represents the rate of deformation, whereas the antisymmetric part represents the local rotation of the material element.

3. Rate of Strain Tensor

The rate of strain tensor is defined as the symmetric part of the velocity gradient tensor and is expressed as

$$\boxed{e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}$$

In three-dimensional Cartesian coordinates, the rate of strain tensor is

$$[e_{ij}] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}.$$

Since the rate of strain tensor is symmetric,

$$\boxed{e_{ij} = e_{ji}}.$$

Therefore, it has six independent components in three dimensions.

4. Physical Significance of the Components

The diagonal components of the rate of strain tensor represent the normal rates of extension or compression along the coordinate directions and are given by

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z}.$$

The off-diagonal components represent the rates of angular deformation. For example,

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

The corresponding engineering shear strain rate is $2e_{xy}$. Similar interpretations hold for the other shear components.

5. Principal Rates of Strain

The principal rates of strain are the maximum and minimum normal rates of deformation occurring along mutually perpendicular directions at a given point, where the shear rate of strain vanishes.

They are obtained by calculating the eigenvalues of the rate of strain tensor. For a two-dimensional rate of strain tensor,

$$[e_{ij}] = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{bmatrix},$$

the principal rates of strain are determined from the characteristic equation

$$\boxed{|e_{ij} - \lambda I| = 0}$$

or equivalently,

$$\begin{vmatrix} e_{xx} - \lambda & e_{xy} \\ e_{xy} & e_{yy} - \lambda \end{vmatrix} = 0.$$

On expanding the determinant, we obtain

$$\lambda^2 - (e_{xx} + e_{yy})\lambda + (e_{xx}e_{yy} - e_{xy}^2) = 0.$$

The roots of this equation give the two principal rates of strain.

Ex. For the two-dimensional rate-of-strain tensor $[e_{ij}] = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, find the principal rates of strain.

Sol: Comparing the given tensor with the general two-dimensional rate of strain tensor,

$$[e_{ij}] = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{bmatrix},$$

we obtain

$$e_{xx} = 2, \qquad e_{xy} = 1, \qquad e_{yy} = 3.$$

The principal rates of strain are the eigenvalues of the given tensor. Therefore, the characteristic equation is

$$|e_{ij} - \lambda I| = 0.$$

Substituting the given values, we get

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

$$\implies (2-\lambda)(3-\lambda) - 1 = 0.$$

$$\implies 6 - 2\lambda - 3\lambda + \lambda^2 - 1 = 0.$$

Hence,

$$\boxed{\lambda^2 - 5\lambda + 5 = 0}.$$

Solving,

$$\lambda = \frac{5 \pm \sqrt{5}}{2}.$$

Thus, the two principal rates of strain are

$$\boxed{\lambda_1 = \frac{5+\sqrt{5}}{2}}$$

and

$$\boxed{\lambda_2 = \frac{5 - \sqrt{5}}{2}}.$$

Numerically,

$$\boxed{\lambda_1 = 3.618, \qquad \lambda_2 = 1.382}.$$

The principal directions can be determined using the relation

$$\tan 2\theta = \frac{2e_{xy}}{e_{xx} - e_{yy}}.$$

Substituting the given values,

$$\tan 2\theta = \frac{2(1)}{2 - 3} = -2.$$

Hence,

$$2\theta = \tan^{-1}(-2).$$

The two mutually perpendicular principal directions are

$$\boxed{\theta_1 = 58.28^\circ, \qquad \theta_2 = -31.72^\circ}.$$

These directions correspond respectively to the maximum and minimum principal rates of strain, measured from the positive x-axis.

8. Physical Interpretation

The maximum principal rate of strain is 3.618, indicating the direction of maximum extension rate, whereas the minimum principal rate of strain is 1.382, indicating the direction of minimum extension rate. Along the principal directions, the shear rate of strain vanishes. Since both principal rates of strain are positive, the material element undergoes extension along both principal directions.

9. Final Result

The principal rates of strain for the given two-dimensional rate of strain tensor are

$$\boxed{\begin{aligned} e_1 &= \frac{5+\sqrt{5}}{2} = 3.618, \\ e_2 &= \frac{5-\sqrt{5}}{2} = 1.382. \end{aligned}}$$

The corresponding principal directions are

$$\boxed{\theta_1 = 58.28^\circ, \qquad \theta_2 = -31.72^\circ.}$$