Milne-Thompson Circle Theorem

Dr. Rajshekhar Roy Baruah

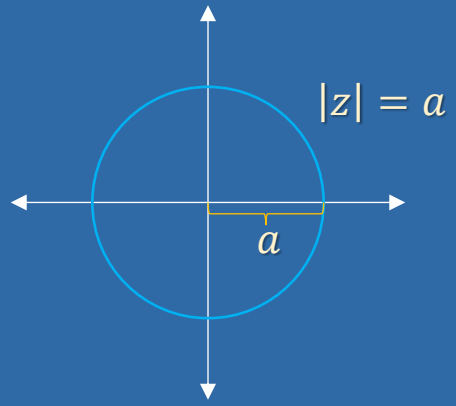
Prerequisite

- Complex Potential
- Singularities
- Conjugate of z
- Properties of $|z|$ viz.

$$z + \bar{z} = real\ quantity$$

Milne-Thomson Circle Theorem

Statement: Let $f(z)$ be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z| = a$. Then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex potential is given by $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ for $|z| \geq a$.



Milne-Thomson Circle Theorem

Statement: Let $f(z)$ be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z| = a$. Then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex potential is given by $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ for $|z| \geq a$.

Proof: Let, C be the cross-section of the cylinder $|z| = a$. Then on C,

$$z\bar{z} = a^2$$

$$\Rightarrow \bar{z} = \frac{a^2}{z}$$

Hence for points on the circle, we have

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = f(z) + \bar{f}(\bar{z})$$

or, $\phi + i\psi = f(z) + \bar{f}(\bar{z})$

or, $\phi + i\psi = f(z) + \overline{f(z)}$ (1)

Since the quantity on R.H.S. of (1) is purely real, equating imaginary parts of (1) we get $\psi = 0$ on C. Hence C is a streamline in the new flow.

By the hypothesis all the singularities of $f(z)$ lie outside of the circle $|z| = a$ and so the singularities of $f\left(\frac{a^2}{z}\right)$ lie inside the circle $|z| = a$.

Hence the singularities of $\bar{f}\left(\frac{a^2}{z}\right)$ also lie inside the circle $|z| = a$. Thus we find that the additional term $\bar{f}\left(\frac{a^2}{z}\right)$ introduces no new singularities into the flow outside the circle $|z| = a$.

Hence $|z| = a$ is a possible boundary for the new flow and $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ is the appropriate complex potential for the new flow.

Theorem of Blasius

Dr. Rajshekhar Roy Baruah

The Theorem of Blasius

Statement: In a steady two-dimensional irrotational motion of an incompressible fluid under no external forces given by the complex potential $w=f(z)$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force (X,Y) and a couple of moment M about the origin of coordinates, then

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz$$

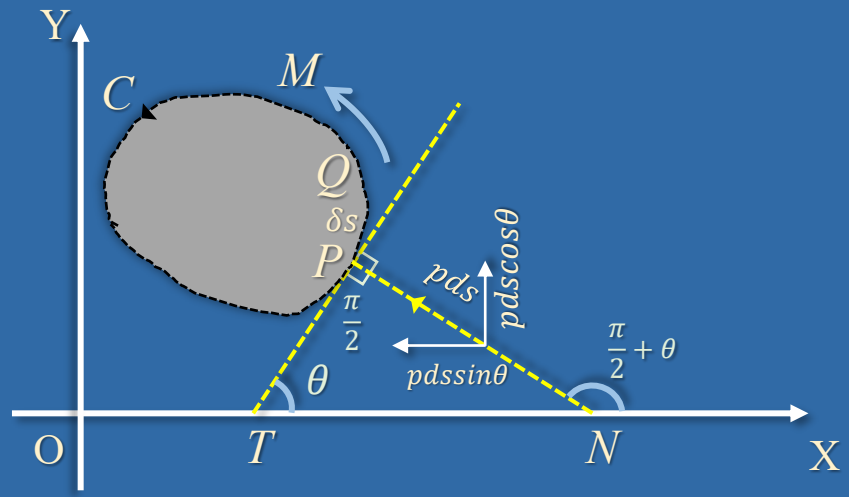
$$M = Real\ part\ of \left\{ -\frac{1}{2}\rho \int_C z \left(\frac{dw}{dz}\right)^2 dz \right\}$$

where ρ is the fluid density and integrals are taken round the contour C of the cylinder.

Proof: Let us consider a cylinder of any cross section C, placed in steady, irrotational flow of a fluid.

Let $P(x,y)$ be any point on an element δs on the contour C. The tangent at P makes an angle θ with the x-axis.

The fluid thrust at P will act along the inward drawn normal to the the cylinder which gives rise to a force system.



The elemental force components on ds along axis of X and Y are respectively

$$dX = pds\cos\left(\theta + \frac{\pi}{2}\right) = -pds\sin\theta$$

$$dY = pds\sin\left(\theta + \frac{\pi}{2}\right) = +pds\cos\theta$$

where p is the pressure.

Thus the net force acting on this surface element *ds* is

$$dF = dX + idY = p(-\sin\theta + \cos\theta)dS$$

Since

$$z = x + iy$$

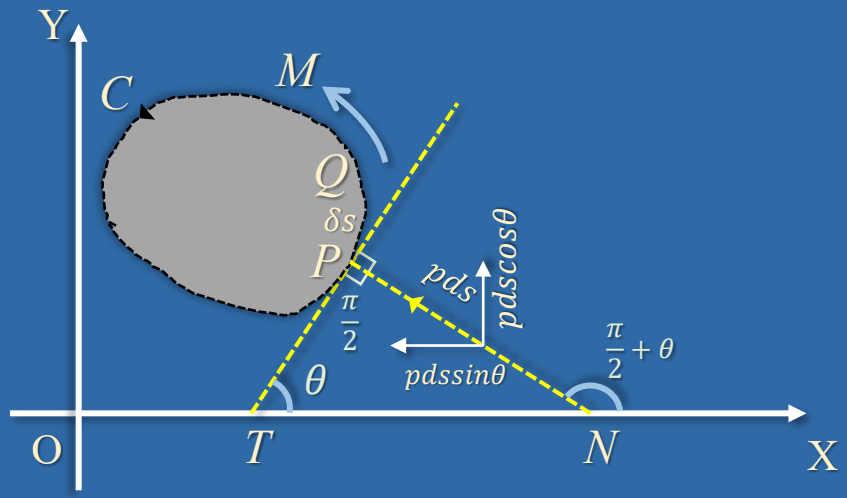
$$\Rightarrow dz = dx + idy$$

$$\Rightarrow dz = \left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds$$

$$\Rightarrow dz = (\cos\theta + i\sin\theta)ds$$

$$\Rightarrow dz = e^{i\theta}ds$$

$$\therefore dF = ip(\cos\theta + i\sin\theta)ds = ipe^{i\theta}ds = ipdz$$



By the Bernoulli's equation, the pressure in a steady, irrotational flow is given by

$$p = p_0 - \frac{1}{2}\rho q^2$$

where p_0 is the pressure at the stagnation point and q is the speed at the point P.

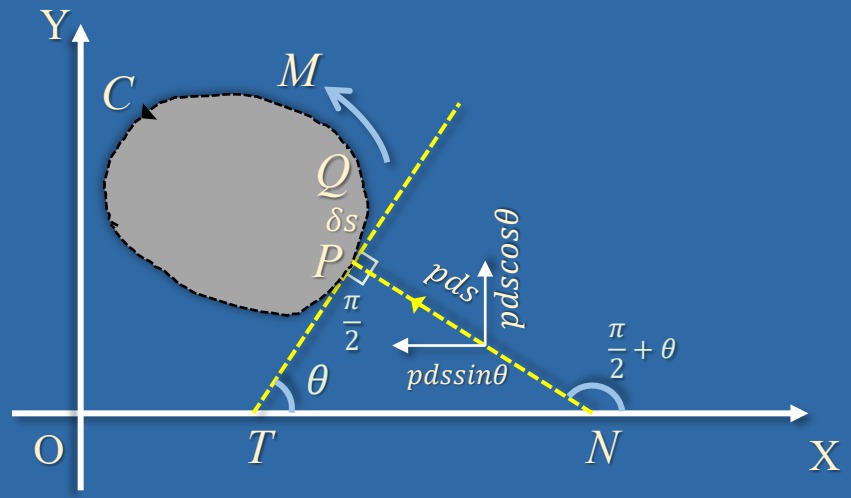
The complex potential is given by

$$w = f(z) => \phi + i\psi = f(x + iy)$$

The velocity potential and the stream function satisfies Cauchy-Riemann equations, therefore

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$(1) => \frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} => \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$



$$\Rightarrow \frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

$$\Rightarrow \frac{dw}{dz} = -u + iv$$

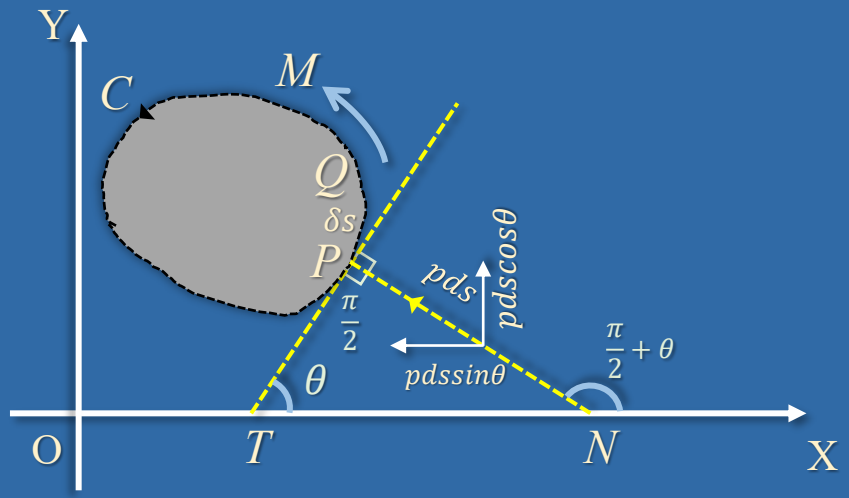
Again on the solid boundary,

$$\frac{dw}{dz} = -u + iv = -q(\cos\theta - \sin\theta)$$

$$\Rightarrow \frac{dw}{dz} = -qe^{-i\theta} \qquad (3)$$

Integrating equation (1) we get,

$$\int_C dF = \int_C ipdz = i\int_C \left(p_0 - \frac{1}{2}\rho q^2\right)dz$$



The first integral vanishes because C is a closed contour.

$$F = -\frac{1}{2}i\rho \int_C q^2 dz = -\frac{1}{2}i\rho \int_C q^2 e^{i\theta} ds$$

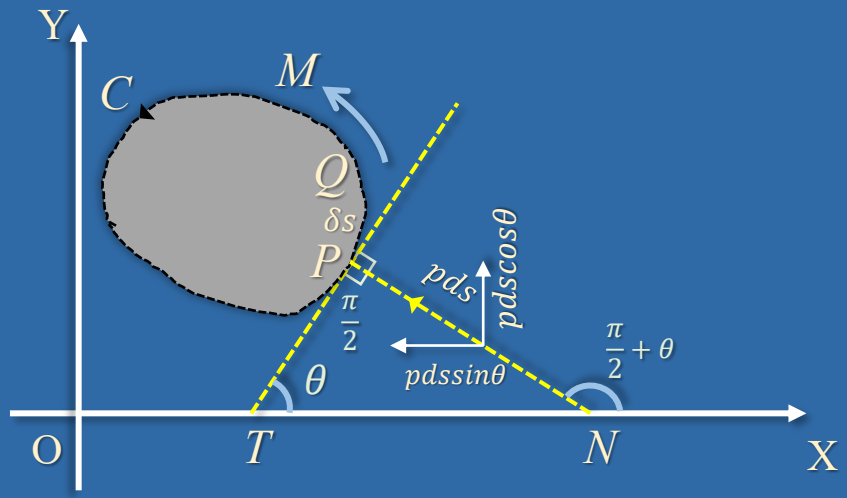
$$=> F = -\frac{1}{2}i\rho \int_C (q^2 e^{i2\theta})(e^{-i\theta} ds)$$

$$=> \bar{F} = X - iY = \frac{1}{2}i\rho \int_C (q^2 e^{-i2\theta})(e^{i\theta} ds)$$

where $\bar{F}$ is the complex conjugate of F.

From (1) and (3) we have,

$$X - iY = \frac{1}{2}i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad (4)$$



The moment about the origin of the fluid thrust acting on the element *ds* is given by

$$dM = (pds\sin\theta)y + (pds\cos\theta)x = p(ydy + xdx)$$

$$\Rightarrow M = \int_C p(ydy + xdx)$$

$$\Rightarrow M = \int_C \left(p_0 - \frac{1}{2}\rho q^2\right)(ydy + xdx)$$

$$\Rightarrow M = p_0 \int_C (ydy + xdx) - \frac{1}{2}\rho \int_C q^2(ydy + xdx)$$

$$\Rightarrow M = -\frac{1}{2}\rho \int_C q^2\left(x\frac{dx}{ds} + y\frac{dy}{ds}\right)ds$$

The first integral vanishes because $(ydy + xdx)$ is an exact differential.

$$\Rightarrow M = -\frac{1}{2}\rho \int_C q^2 (x\cos\theta + y\sin\theta)\, ds$$

$$\Rightarrow M = \text{Real part of } -\frac{1}{2}\rho \int_C q^2 (x+iy)(\cos\theta - i\sin\theta)\, ds$$

$$\Rightarrow M = \text{Real part of } -\frac{1}{2}\rho \int_C q^2 z e^{-i\theta}\, ds$$

$$\Rightarrow M = \text{Real part of } -\frac{1}{2}\rho \int_C z\left(q^2 e^{-i2\theta}\right)\left(e^{i\theta} ds\right)$$

$$\Rightarrow M = \text{Real part of } -\frac{1}{2}\rho \int_C z\left(\frac{dw}{dz}\right)^2 dz$$

where the moment M is defined as the hydrodynamic moment action on the body and taken positive when it acts in the clockwise direction.