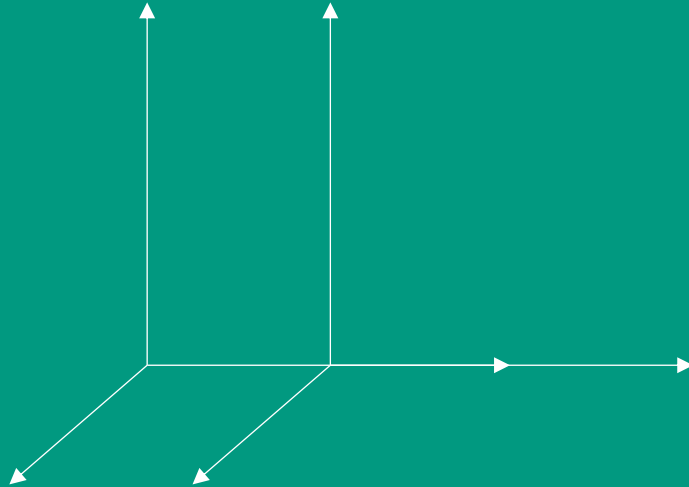


SPECIAL THEORY OF RELATIVITY

MAT 305



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RELATIVISTIC ADDITION LAW OF VELOCITY OR VELOCITY ADDITION THEOREM

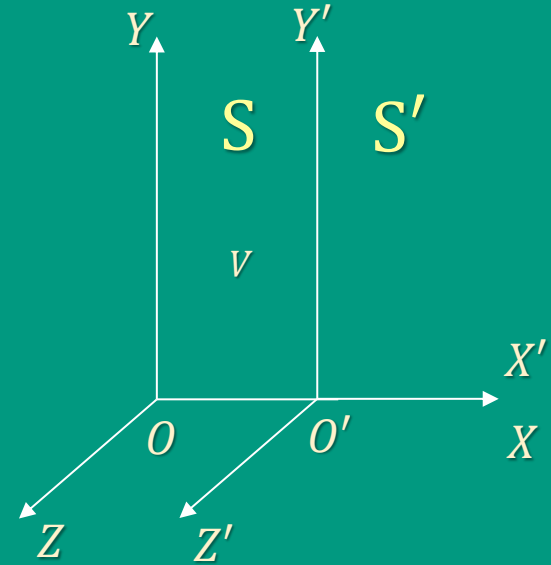
RELATIVISTIC ADDITION LAW OF VELOCITY OR VELOCITY ADDITION THEOREM

Let us consider two inertial frame of references S and S' having cartesian coordinate axes as XYZ and $X'Y'Z'$ and origins O and O' respectively.

At time $t = 0 = t'$, both the frames are at rest so that their origin O and O' coincide with each other.

Let dx, dy, dz be the displacement measured in the S frame in time dt and dx', dy', dz' be the displacement measured in the S' frame in time dt' .

Let (u_x, u_y, u_z) and (u'_x, u'_y, u'_z) be the velocity components of a particle w.r.t. S and S' frame respectively.



$$\therefore \quad u_x = \frac{dx}{dt} \quad u_y = \frac{dy}{dt} \quad u_z = \frac{dz}{dt}$$

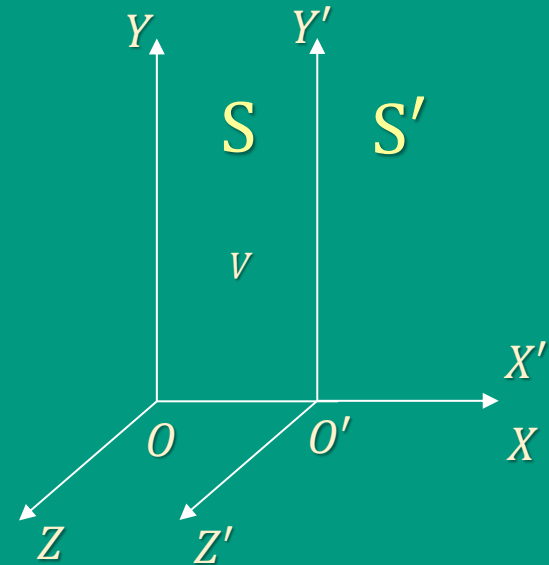
$$\& \quad u'_x = \frac{dx'}{dt'} \quad u'_y = \frac{dy'}{dt'} \quad u'_z = \frac{dz'}{dt'}$$

Inverse Lorentz transformation equations are

$$x = \gamma(x' + vt'), \quad y = y', \quad z = z', \quad t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$

$$dx = \gamma(dx' + vdt'), \quad dy = dy', \quad dz = dz',$$

$$dt = \gamma\left(dt' + \frac{vdx'}{c^2}\right)$$



$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now,

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)}$$

$$\Rightarrow u_x = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2} \frac{dx'}{dt'}}$$

$$\Rightarrow u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{\frac{dy'}{dt}}{\gamma\left(1 + \frac{v}{c^2} \frac{dx'}{dt'}\right)}$$

$$\Rightarrow u_y = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{\frac{dz'}{dt}}{\gamma\left(1 + \frac{v}{c^2} \frac{dx'}{dt'}\right)}$$

$$\Rightarrow u_z = \frac{u'_z}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

Thus,

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

DEDUCTIONS

(i) Let $u'_x = u$, $u'_y = 0$, $u'_z = 0$ then $u_x = \frac{u + v}{1 + \frac{uv}{c^2}}$

This is the *relativistic law of addition of two velocities* in the same direction i.e. x-axis.

(ii) The velocity of light is an *absolute constant* independent of the motion of the reference system.

Let $u = c$ then we get, $u_x = \frac{c + v}{1 + \frac{cv}{c^2}} = \frac{c + v}{1 + \frac{v}{c}} = \frac{c(c + v)}{c + v} = c$

This shows that no material particle can *exceed the velocity of light*. Otherwise c is upper limit of the velocity of material particle. The velocity of light c cannot be changed by adding to it a velocity smaller than c .

(iii) The addition of two velocities each of which is less than c is also less than c .

Let $u = c - \lambda$, $v = c - \mu$ where λ, μ are positive constants.

By relativistic addition law of velocities,

$$u_x = \frac{c - \lambda + c - \mu}{1 + \frac{(c - \lambda)(c - \mu)}{c^2}} = \frac{c - \lambda + c - \mu}{1 + \frac{c^2 - c(\lambda + \mu) + \lambda\mu}{c^2}} = \frac{2c - \lambda - \mu}{\frac{2c^2 - c(\lambda + \mu) + \lambda\mu}{c^2}} = \frac{c[2c - (\lambda + \mu)]}{2c - (\lambda + \mu) + \frac{\lambda\mu}{c}}$$

$$\lambda > 0, \mu > 0 \Rightarrow \frac{\lambda\mu}{c} > 0 \Rightarrow 2c - (\lambda + \mu) + \frac{\lambda\mu}{c} > 2c - (\lambda + \mu)$$

$$\Rightarrow \frac{2c - (\lambda + \mu)}{2c - (\lambda + \mu) + \frac{\lambda\mu}{c}} < 1$$

$$\therefore u_x < c$$

This shows that the addition of two velocities each of which is less than c is also less than c .

Important points on velocity of light –

- ❑ It is constant in all directions.
- ❑ It is the same for all observers irrespective of the velocities of the source and the observer.
- ❑ It is invariant for any two systems joined by Lorentz transformation.
- ❑ The addition of any velocity to the velocity of light is simply equal to c .
- ❑ It is impossible to send out signals with a velocity greater than c .

Transformation of velocity:

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u_y = \frac{u'_y}{\gamma \left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u_z = \frac{u'_z}{\gamma \left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2} u_x}$$

$$u'_y = \frac{u_y}{\gamma \left(1 - \frac{v}{c^2} u_x\right)}$$

$$u'_z = \frac{u_z}{\gamma \left(1 - \frac{v}{c^2} u_x\right)}$$

Transformation of Acceleration:

By velocity transformation we have,

$$\left. \begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \\ u'_y &= \frac{u_y}{\gamma \left(1 - \frac{v}{c^2} u_x\right)} \\ u'_z &= \frac{u_z}{\gamma \left(1 - \frac{v}{c^2} u_x\right)} \end{aligned} \right\} \quad (1)$$

By Lorentz transformation we have,

$$x' = \gamma(x - vt) \quad y' = y, \quad z' = z, \quad t' = \gamma \left(t - \frac{xv}{c^2} \right) \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$dt' = \gamma \left(dt - \frac{v dx}{c^2} \right) = \gamma dt \left(1 - \frac{v}{c^2} u_x \right)$$

$$\Rightarrow dt = \frac{dt'}{\gamma \left(1 - \frac{v}{c^2} u_x \right)} \quad (2)$$

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \Rightarrow \frac{du'_x}{dt} = \frac{d}{dt} \left(\frac{u_x - v}{1 - \frac{v}{c^2} u_x} \right) \Rightarrow \frac{du'_x}{dt} = \frac{d}{du_x} \left(\frac{u_x - v}{1 - \frac{v}{c^2} u_x} \right) \frac{du_x}{dt}$$

$$\Rightarrow a'_x = \frac{\left(1 - \frac{v^2}{c^2} \right)^{3/2}}{\left(1 - \frac{v}{c^2} u_x \right)^3} \cdot a_x$$

Q. Show that four dimensional volume element $dx dy dz dt$ is invariant under Lorentz transformations.

Q. Show that

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

is invariant under Lorentz transformations.

Q. Show that

$$x^2 + y^2 + z^2 - c^2 t^2$$

is invariant under Lorentz transformations.

Q. Show that

$$E^2 - P^2 c^2 = m^2 c^4$$

is invariant under Lorentz transformations.

Thank You