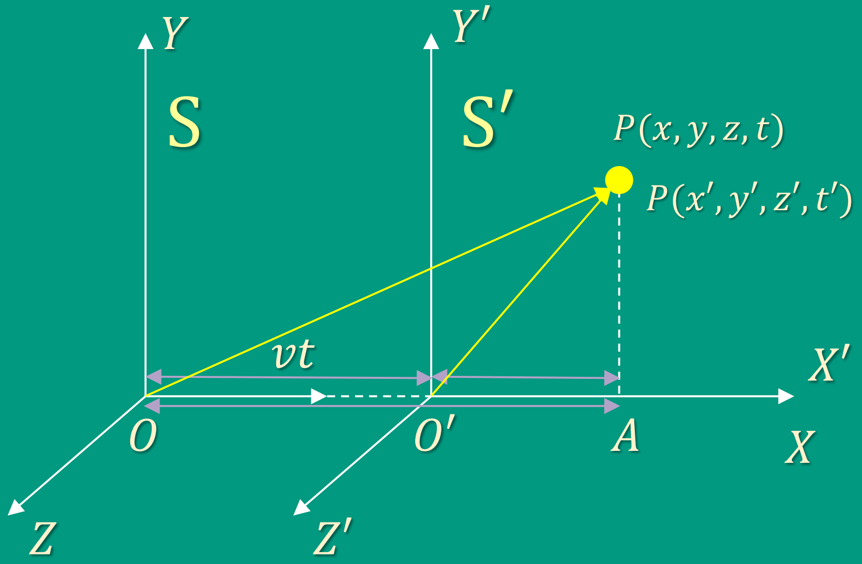
SPECIAL THEORY OF RELATIVITY

MATSPL25024B

LORENTZ TRANSFORMATION

Let in the course of motion of S' relative to S in the x-direction with a speed v a flash of light is emitted at the origins of S and S' when they were just coincident.



Let us identify this instant as $t = 0 = t'$.

For each observer, the flash of light would spread out as a spherical wave with speed c according to the postulate of relativity.

$$x^2 + y^2 + z^2 = d^2$$

$$=> x^2 + y^2 + z^2 = c^2t^2 \quad (1)$$

Similarly,

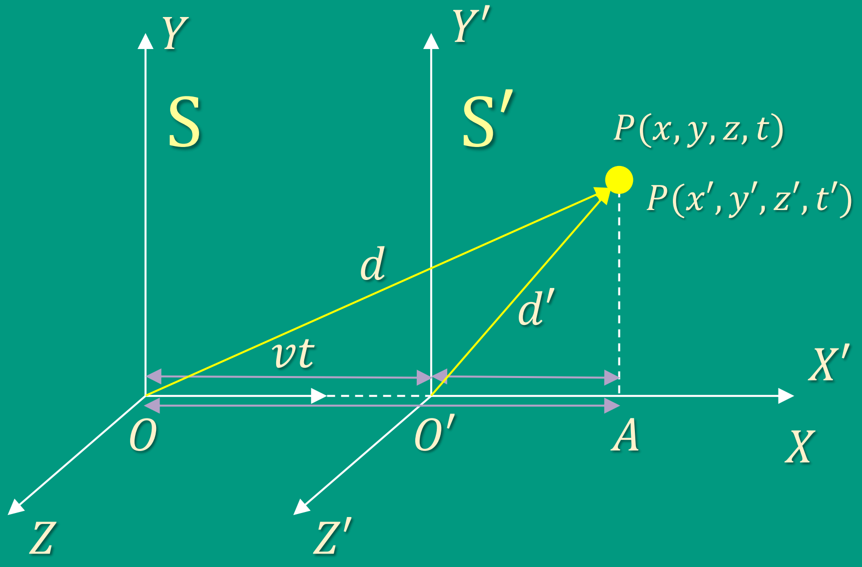
$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad (2)$$

Since the relative motion occurs in x-direction, the transverse coordinates should remain unchanged i.e.

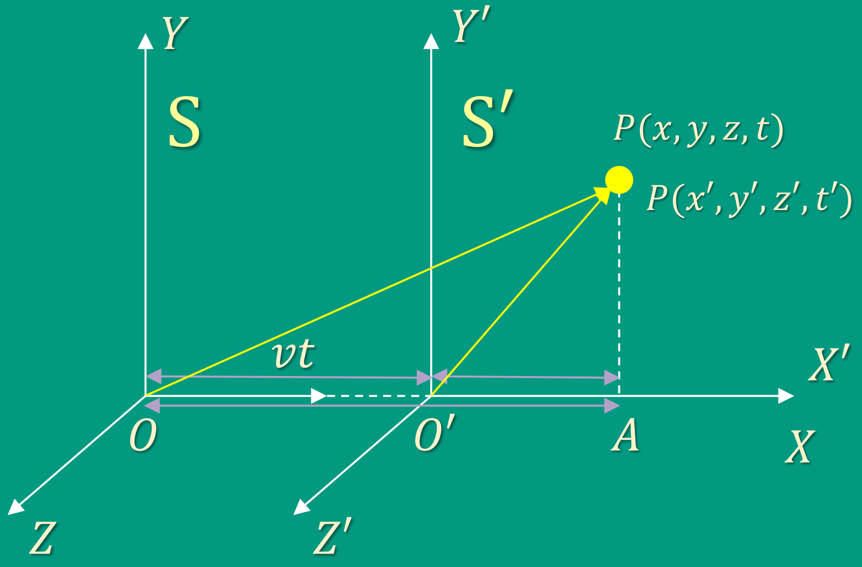
$$y = y', \quad z = z'$$

(1)-(2)=>

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad (3)$$



Now the transformation law should be such that at any instant, at O', x' should be equal to zero, but x should be equal to vt. On the other hand, at the same instant, at O, x should be equal to zero, but x' should be equal to $-vt'$.



Keeping these conditions in view, the simplest transformation is

$$x' = k(x - vt) \tag{4}$$

$$x = k'(x' + vt') \tag{5}$$

where k and k' are constants.

Eliminating x' from equations (4) and (5) we get,

$$x = k'[k(x - vt) + vt']$$

$$=> x = kk'(x - vt) + k'vt'$$

$$=> x - kk'(x - vt) = k'vt'$$

$$=> t' = \frac{x - kk'(x - vt)}{k'v}$$

$$=> t' = \frac{x}{k'v} - \frac{kk'x}{k'v} + \frac{kk'vt}{k'v}$$

$$=> t' = kt - \frac{kx}{v} + \frac{x}{k'v}$$

$$=> t' = kt - \frac{x}{v}\left(k - \frac{1}{k'}\right)$$

$$=> t' = k\left[t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right] \qquad (6)$$

From equation (3) we get,

$$x^2 - c^2t^2 = x'^2 - c^2t'^2$$

$$=> x^2 - c^2t^2 = \{k(x - vt)\}^2 - c^2\left[k\left\{t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right\}\right]^2$$

$$=> x^2 - c^2t^2 = k^2(x - vt)^2 - c^2k^2\left[t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right]^2$$

$$=> x^2 - c^2t^2 = k^2(x^2 + v^2t^2 - 2xvt) - c^2k^2$$

$$\left[t^2 + \frac{x^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2 - 2.t.\frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right]$$

$$=> x^2 - c^2t^2 = k^2x^2 + k^2v^2t^2 - 2k^2xvt - c^2k^2t^2 - \frac{c^2k^2x^2}{v^2}$$

$$\left(1 - \frac{1}{kk'}\right)^2 + \frac{2c^2k^2xt}{v}\left(1 - \frac{1}{kk'}\right)$$

$$=> x^2 - c^2t^2 = k^2x^2 + k^2v^2t^2 - 2k^2xvt - c^2k^2t^2 - \frac{c^2k^2x^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2 + \frac{2c^2k^2xt}{v}\left(1 - \frac{1}{kk'}\right)$$

$$=> x^2\left[1 - k^2 + \frac{c^2k^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2\right]$$

$$=> x^2 - c^2t^2 = k^2x^2 + k^2v^2t^2 - 2k^2xvt - c^2k^2t^2 - \frac{c^2k^2x^2}{v^2}$$

$$\left(1 - \frac{1}{kk'}\right)^2 + \frac{2c^2k^2xt}{v}\left(1 - \frac{1}{kk'}\right)$$

$$=> x^2\left[1 - k^2 + \frac{c^2k^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2\right] + 2xt\left[k^2v - \frac{c^2k^2}{v}\left(1 - \frac{1}{kk'}\right)\right]$$

$$=> x^2 - c^2t^2 = k^2x^2 + k^2v^2t^2 - 2k^2xvt - c^2k^2t^2 - \frac{c^2k^2x^2}{v^2}\left(1-\frac{1}{kk'}\right)^2 + \frac{2c^2k^2xt}{v}\left(1-\frac{1}{kk'}\right)$$

$$=> x^2\left[1 - k^2 + \frac{c^2k^2}{v^2}\left(1-\frac{1}{kk'}\right)^2\right] + 2xt\left[k^2v - \frac{c^2k^2}{v}\left(1-\frac{1}{kk'}\right)\right] - c^2t^2 - k^2v^2t^2 + c^2k^2t^2 = 0$$

$$=> x^2 - c^2t^2 = k^2x^2 + k^2v^2t^2 - 2k^2xvt - c^2k^2t^2 - \frac{c^2k^2x^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2 + \frac{2c^2k^2xt}{v}\left(1 - \frac{1}{kk'}\right)$$

$$=> x^2\left[1 - k^2 + \frac{c^2k^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2\right] + 2xt\left[k^2v - \frac{c^2k^2}{v}\left(1 - \frac{1}{kk'}\right)\right] - t^2(c^2 + k^2v^2 - c^2k^2) = 0$$

Since this is an identity, this must be true for all values of x and t. Equating the coefficients of x^2, t^2 and xt on both sides we get,

$$c^2 + k^2v^2 - c^2k^2 = 0$$

$$=> c^2 - k^2(c^2 - v^2) = 0$$

$$=> k^2(c^2 - v^2) = c^2$$

$$=> k^2 = \frac{c^2}{c^2 - v^2}$$

$$=> k^2 = \frac{c^2}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

$$=> k^2 = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$=> k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Again,

$$k^2 v - \frac{c^2 k^2}{v}\left(1 - \frac{1}{kk'}\right) = 0$$

$$=> \frac{v}{\left(1 - \frac{v^2}{c^2}\right)} - \frac{c^2}{v\left(1 - \frac{v^2}{c^2}\right)}\left(1 - \frac{1}{kk'}\right) = 0$$

$$=> \frac{c^2}{v\cancel{\left(1 - \frac{v^2}{c^2}\right)}}\left(1 - \frac{1}{kk'}\right) = \frac{v}{\cancel{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$=> \frac{c^2}{v}\left(1 - \frac{1}{kk'}\right) = v$$

$$=> 1 - \frac{1}{kk'} = \frac{v^2}{c^2}$$

$$=> \frac{1}{kk'} = 1 - \frac{v^2}{c^2}$$

$$=> \frac{1}{k'} = k\left(1 - \frac{v^2}{c^2}\right)$$

$$=> \frac{1}{k'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(1 - \frac{v^2}{c^2}\right)$$

$$=> \frac{1}{k'} = \sqrt{1 - \frac{v^2}{c^2}}$$

$$=> k' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore k = k' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting the values of k and k' in equations (3) and (4) we get,

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

From equation (6) we get,

$$t' = k\left[t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right]$$

$$=> t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left[t - \frac{x}{v}\left(1 - \frac{1}{\frac{1}{1 - \frac{v^2}{c^2}}}\right)\right]$$

$$=> t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left[t - \frac{x}{v}\left(1 - 1 + \frac{v^2}{c^2}\right)\right]$$

$$\Rightarrow t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[t-\frac{xv}{c^2}\right] = \frac{t-\frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Thus Lorentz transformation equations are

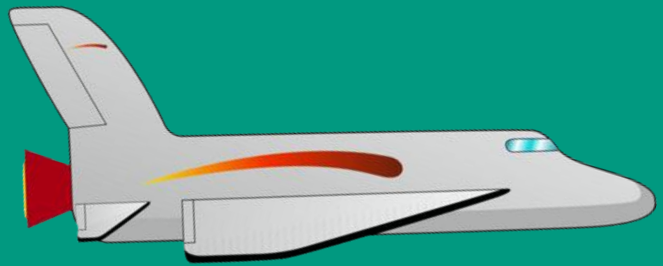
$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \;\; z' = z, \quad t' = \frac{t-\frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The reverse Lorentz transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t = \frac{t' + \frac{x'v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note: When v is much much less than c then Lorentz transformation reduces to Galilean transformation.

CONSEQUENCES OF LORENTZ TRANSFORMATION

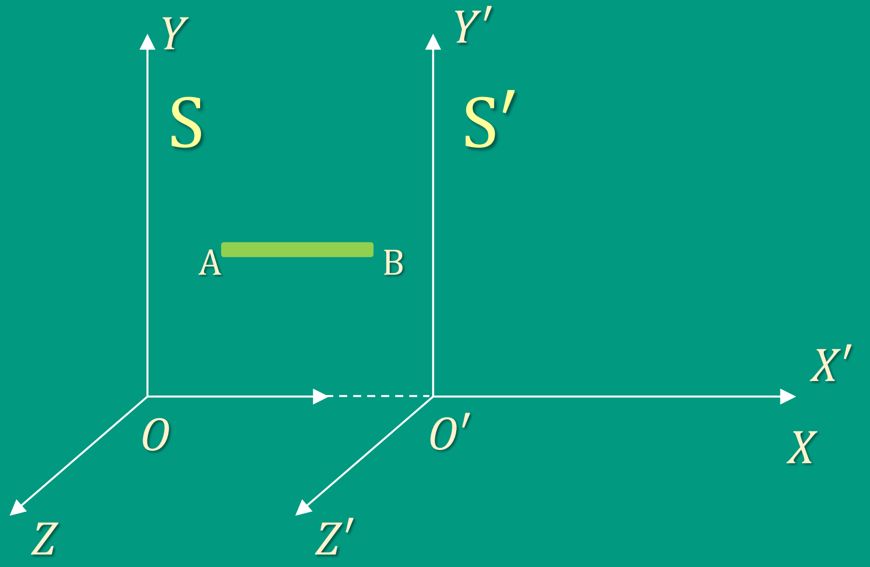


CONSEQUENCES OF LORENTZ TRANSFORMATION

Length Contraction (Lorentz-FitzGerald Contraction):

Let us consider two inertial frame of references S and S' having cartesian coordinate axes as XYZ and $X'Y'Z'$ and origins O and O' respectively.

At time $t = 0 = t'$, both the frames are at rest so that their origin O and O' coincide with each other.



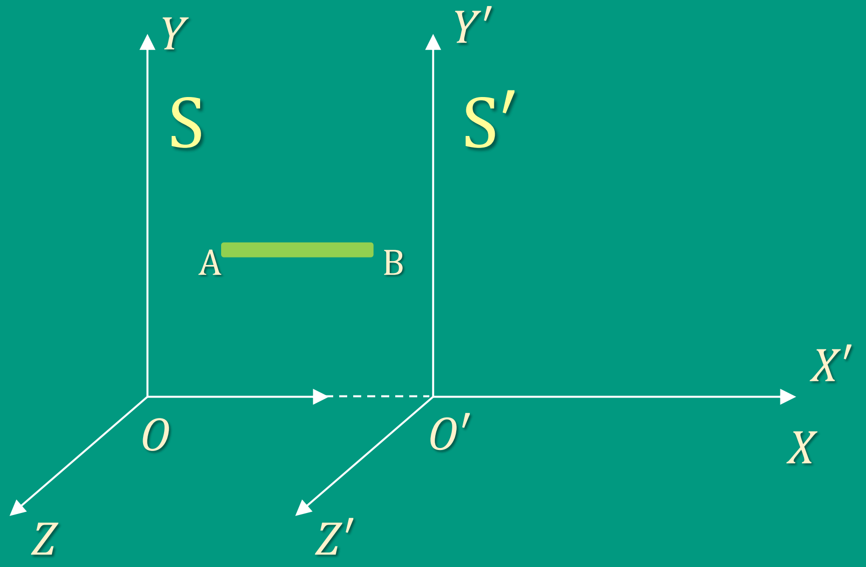
Let a rod AB of length l be placed in S frame parallel to x-axis.

The coordinates of A and B be x_1 and x_2 respectively.

$$\therefore \qquad l = x_2 - x_1 \qquad (1)$$

Let x'_1 and x'_2 be the corresponding coordinate of the rod w.r.t. S' frame . If l' be the length of the rod in S' frame then

$$l' = x'_2 - x'_1 \qquad (2)$$



Now,

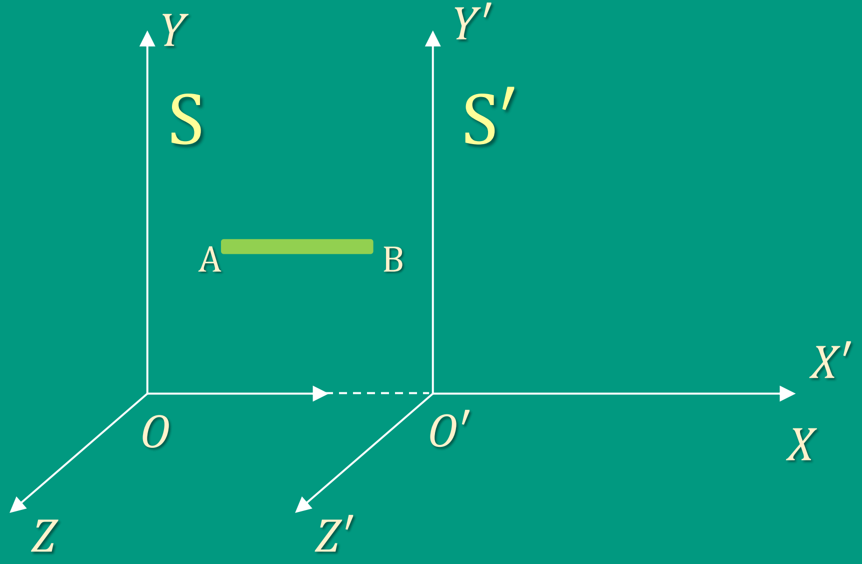
$$l = x_2 - x_1$$

$$= \frac{x_2' + vt_2'}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1' + vt_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

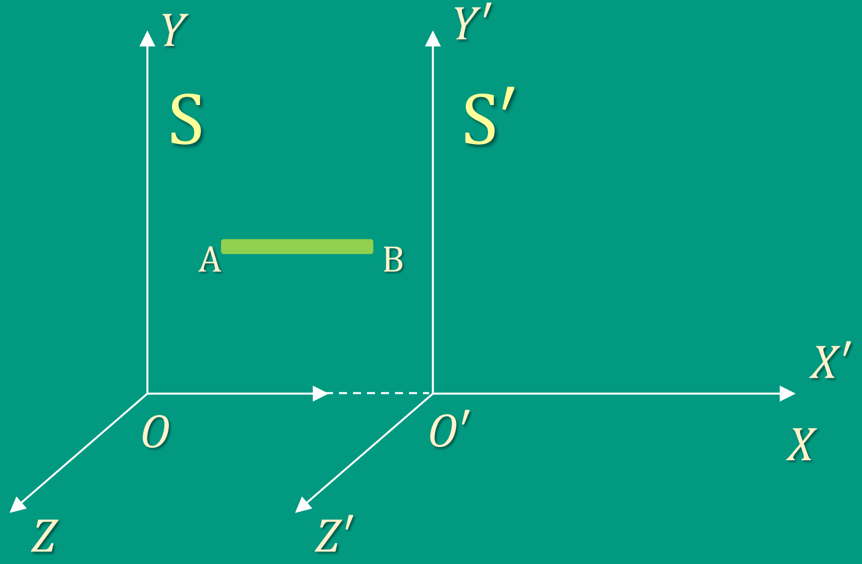
$$= \gamma[x_2' + vt_2' - x_1' - vt_1']$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$= \gamma(x_2' - x_1') + v\gamma(t_2' - t_1') \qquad (3)$$



Let us assume that both the ends of the rod have been measured *simultaneously* by an observer in S' frame (i.e. $t_1' = t_2'$).



Then from equation (3) we get,

$$l = \gamma(x_2' - x_1')$$

$$=> l = \gamma l' \qquad (4)$$

Now,

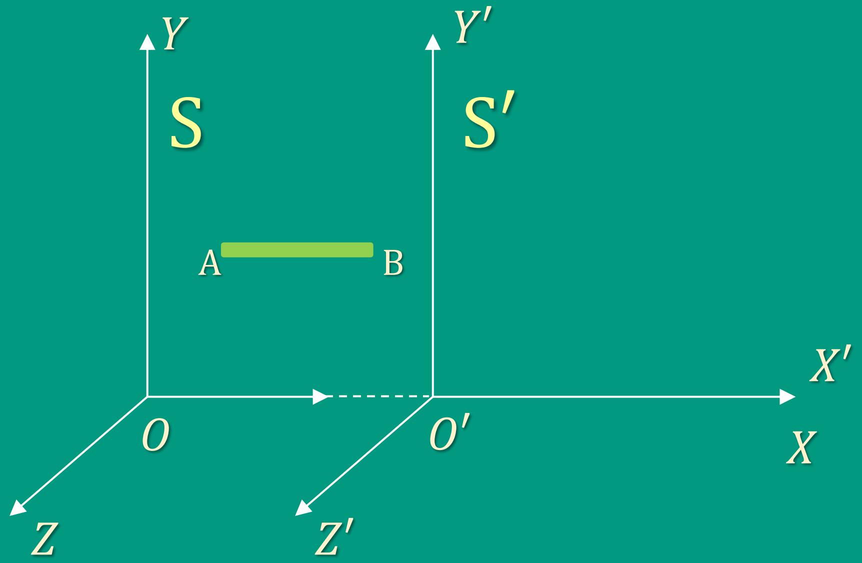
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{v^2}{2c^2} + \cdots$$

$\therefore \quad \gamma > 1$

Then from equation (4) we get,

$$l > l'$$

This shows that the apparent (observed) length of the rod is found to be *contracted* than its actual length when it is placed in the direction of motion.

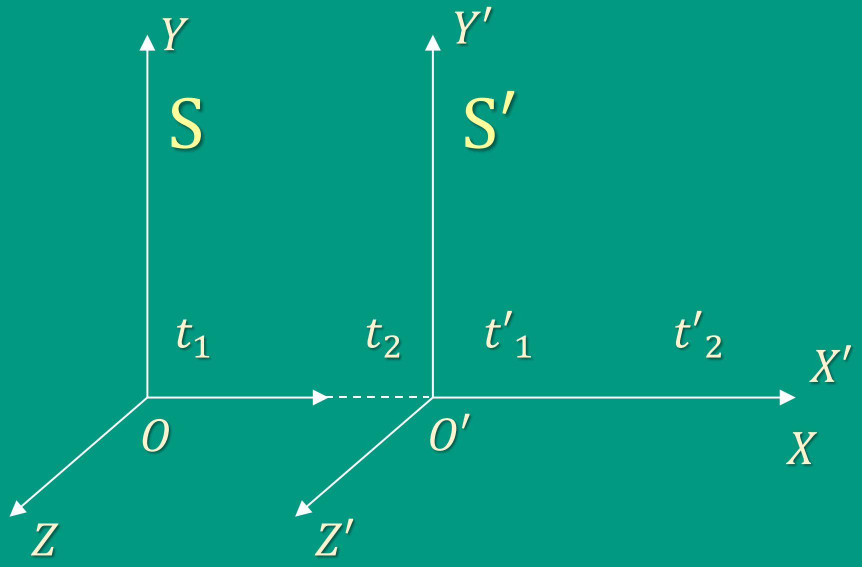


Every rigid body *appears to be longest when at rest* relative to the observer. If the body moves with uniform velocity v relative to the observer then its apparent length is contracted by $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ in the direction of relative motion.

Time Dilation:

Let us consider two inertial frame of references S and S' having cartesian coordinate axes as XYZ and $X'Y'Z'$ and origins O and O' respectively.

At time $t = 0 = t'$, both the frames are at rest so that their origin O and O' coincide with each other.



Let t_1 and t_2 be the time recorded by a clock for two events in S frame.

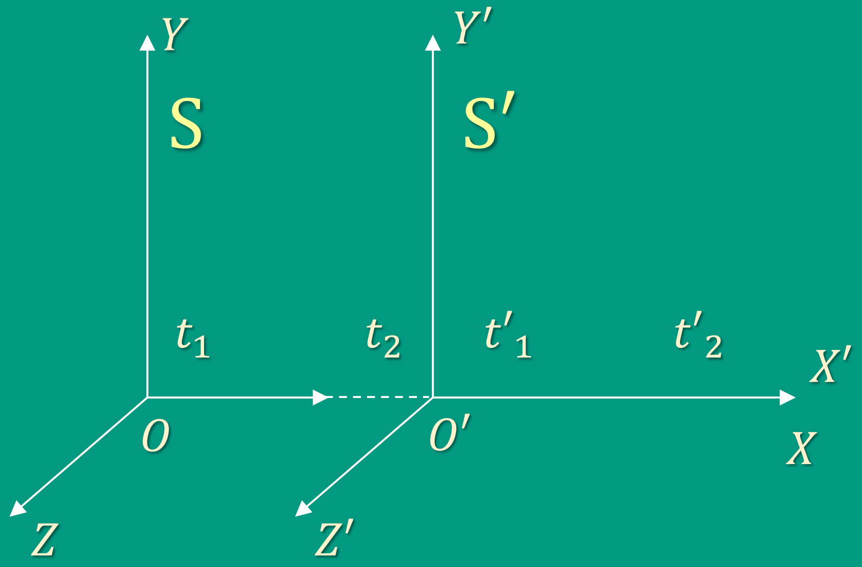
Let t'_1 and t'_2 be the corresponding two times w.r.t. the observer in S' frame.

The time interval for S and S' are $\Delta t = t_2 - t_1$, $\Delta t' = t'_2 - t'_1$

But, $\Delta t' = t'_2 - t'_1$

$$= \frac{t_2 - \frac{x_2 v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1 - \frac{x_1 v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \gamma\left(t_2 - \frac{x_2 v}{c^2}\right) - \gamma\left(t_1 - \frac{x_1 v}{c^2}\right)$$



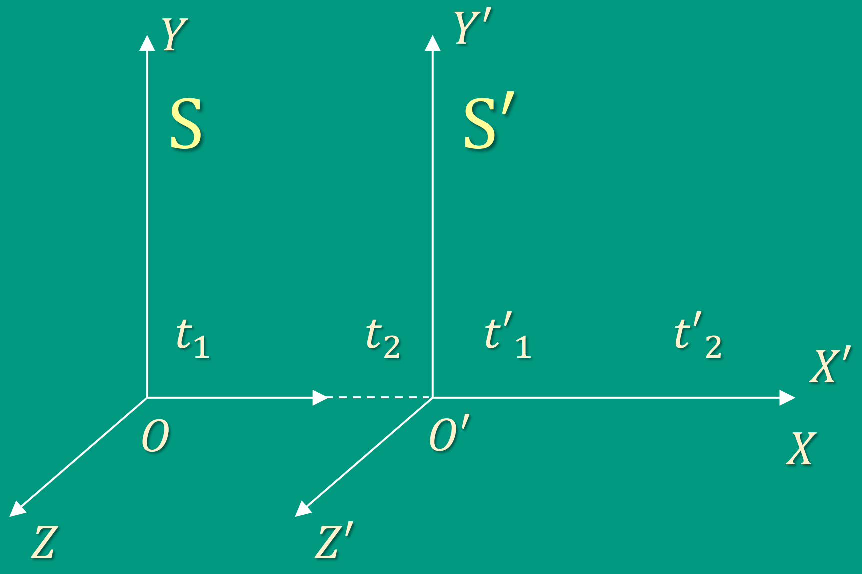
$$= \gamma(t_2 - t_1) - \gamma \frac{v}{c^2}(x_2 - x_1)$$

Since time is recorded from the same position, so $x_1 = x_2$

$$\Delta t' = \gamma(t_2 - t_1)$$

$$\Delta t' = \gamma \Delta t$$

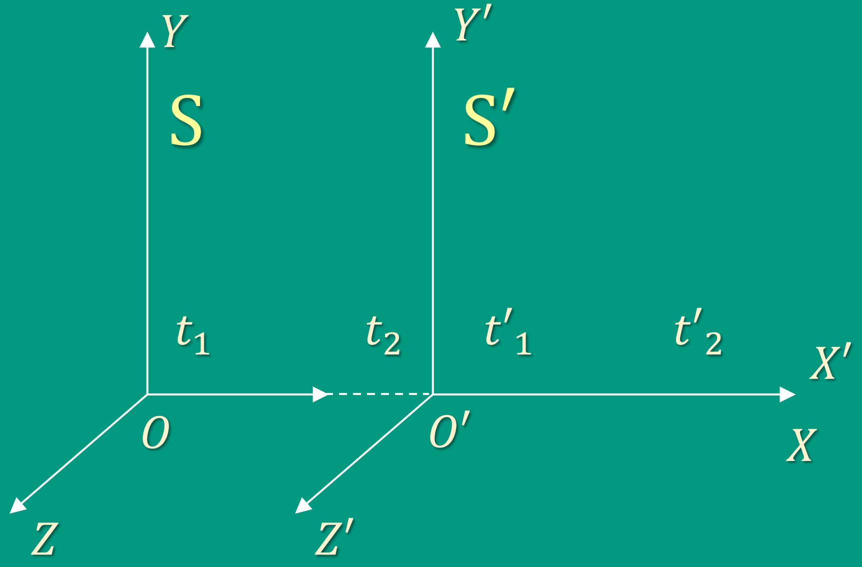
$$\Delta t' > \Delta t$$



$$\left[\because \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{v^2}{2c^2} + \cdots \right]$$

This shows that the time interval of two events is found to be longer w.r.t. the observer in motion.

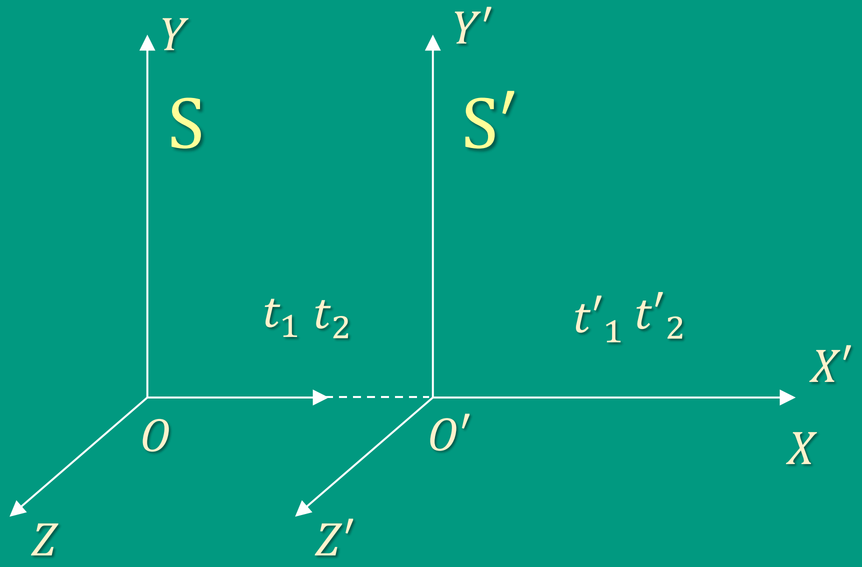
Thus every clock goes slow when it is in motion relative to the observer.



Simultaneity of Two Events:

Let us consider two inertial frame of references S and S' having cartesian coordinate axes as XYZ and $X'Y'Z'$ and origins O and O' respectively.

At time $t = 0 = t'$, both the frames are at rest so that their origin O and O' coincide with each other.



Let two events occur simultaneously i.e. $t_1 = t_2$ but at two different positions x_1 and x_2.

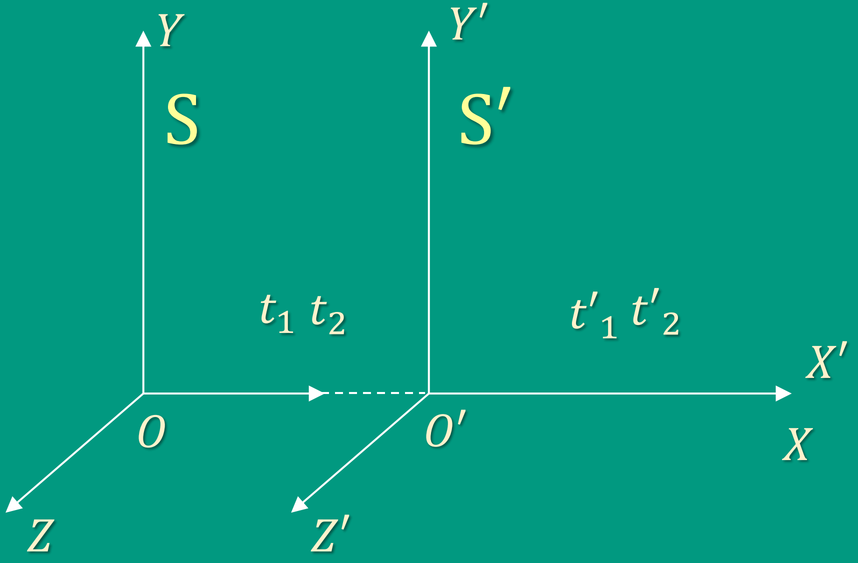
$$t'_2 - t'_1 = \frac{t_2 - \frac{x_2 v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1 - \frac{x_1 v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \gamma\left(t_2 - \frac{x_2 v}{c^2}\right) - \gamma\left(t_1 - \frac{x_1 v}{c^2}\right)$$

$$= \gamma(t_2 - t_1) - \gamma\frac{v}{c^2}(x_2 - x_1)$$

$$= -\gamma\frac{v}{c^2}(x_2 - x_1)$$

$$\neq 0 \qquad [\because x_1 \neq x_2]$$



$$\therefore t'_1 \neq t'_2$$

Hence simultaneity of events in one frame of reference doesnot imply simultaneity in other frame of reference.

Q. The length of a spaceship is 100 metres on the ground. When it is in flight its length observed on the ground is 99 metres. Calculate its speed.

Q. The length of a spaceship is 100 metres on the ground. When it is in flight its length observed on the ground is 99 metres. Calculate its speed.

Sol: Length of spaceship on the ground = 100 m

Length of spaceship when it is in flight = 99 m

Length contraction is given by

$$l' = l\sqrt{1 - \frac{v^2}{c^2}}$$

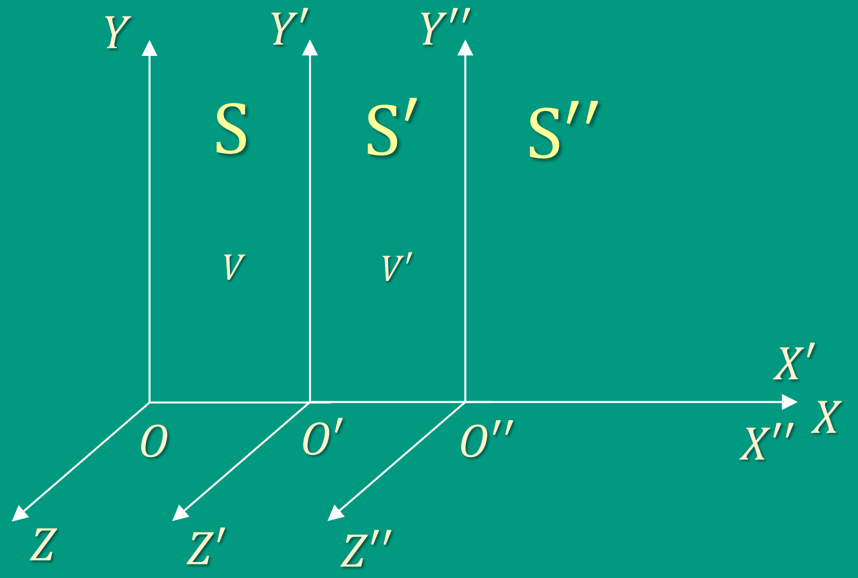
$c = 3 \times 10^8 \text{m/s}$

$v = 42.3 \times 10^6 \text{m/s}$

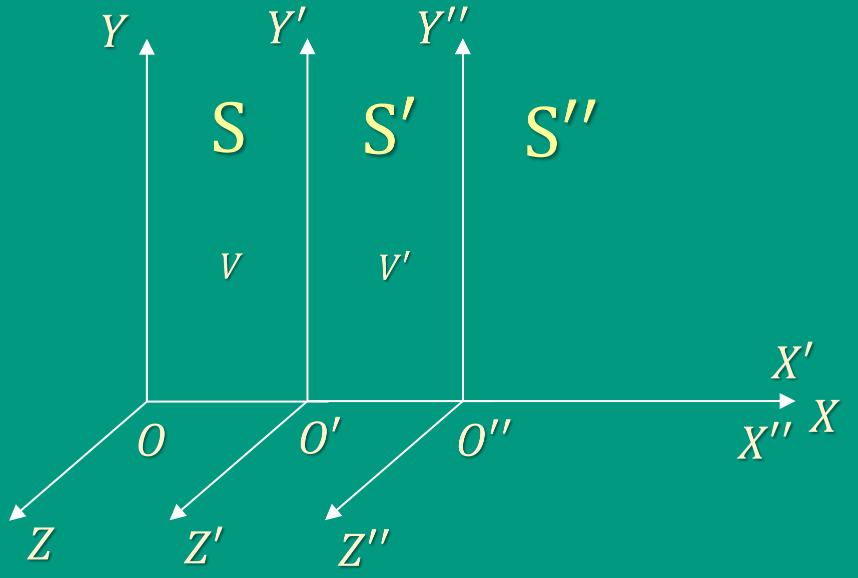
LORENTZ TRANSFORMATION FORMS A GROUP

Let us consider three inertial frame of references S and S' and S'' having cartesian coordinate axes as XYZ, $X'Y'Z'$ and $X''Y''Z''$ and origins O, O' and O'' respectively.

At time $t = t' = t'' = 0$, all the frames are at rest so that their origins O, O' and O'' coincide with each other.



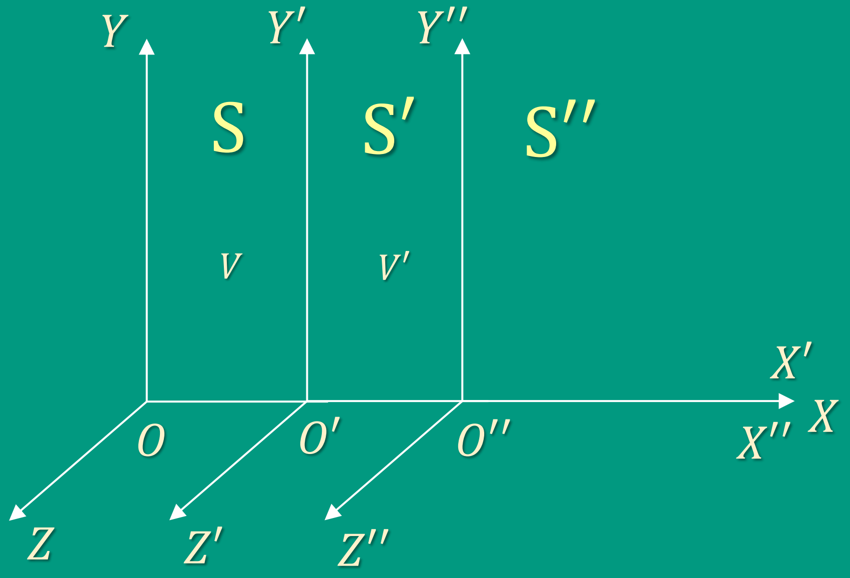
Let us consider a Lorentz transformation from S to S' system where S' moves away from S with velocity v. Then we have



$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y, \quad z' = z, \\ t' &= \gamma\left(t - \frac{xv}{c^2}\right) \end{aligned}\right\} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (1)$$

Let us now consider a successeive Lorentz transformation from S' to S'' system where S'' moves away from S' with velocity v' relative to S'. Then we have

$$\left.\begin{aligned} x'' &= \gamma'(x' - v't') \\ y'' &= y', z'' = z' \\ t'' &= \gamma'\left(t' - \frac{x'v'}{c^2}\right) \end{aligned}\right\} \quad \gamma' = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}} \qquad (2)$$



Using (1) in (2) we get,

$$x'' = \gamma'\left[\gamma(x - vt) - v'\gamma\left(t - \frac{xv}{c^2}\right)\right]$$

$$= \gamma\gamma'\left[\left(1 + \frac{vv'}{c^2}\right)x - (v + v')t\right]$$

$$= \gamma\gamma'\left(1 + \frac{vv'}{c^2}\right)\left[x - \left(\frac{v + v'}{1 + \frac{vv'}{c^2}}\right)t\right] \qquad (3)$$

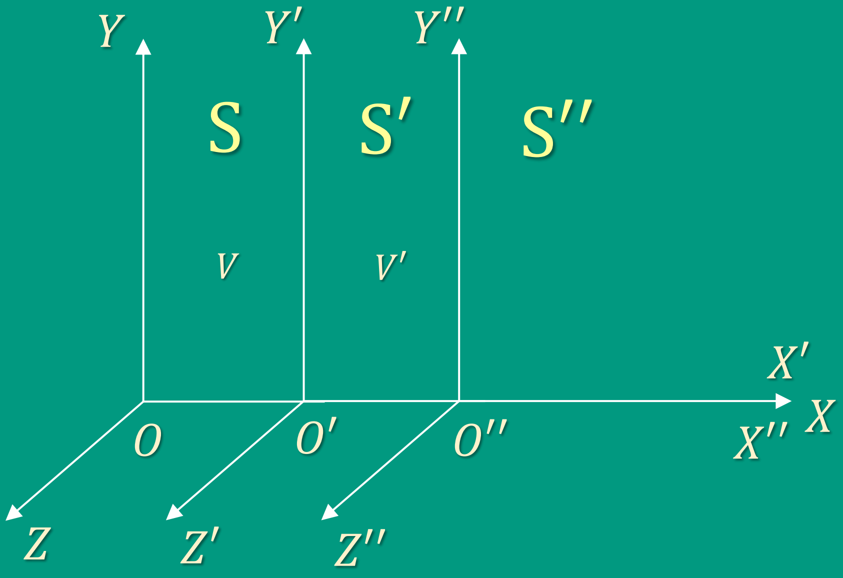
and

$$t'' = \gamma'\left(t' - \frac{x'v'}{c^2}\right)$$

$$= \gamma'\left[\gamma\left(t - \frac{xv}{c^2}\right) - \frac{v'}{c^2}\gamma(x - vt)\right]$$

$$= \gamma\gamma'\left[\left(1 + \frac{vv'}{c^2}\right)t - \left(\frac{v+v'}{c^2}\right)x\right]$$

$$= \gamma\gamma'\left(1 + \frac{vv'}{c^2}\right)\left[t - \frac{v+v'}{c^2}\left(\frac{1}{1 + \frac{vv'}{c^2}}\right)x\right] \tag{4}$$



If v'' be the resultant velocity of the velocities v and v' w.r.t. to S frame then by addition law of velocity,

$$v'' = \frac{v+v'}{1 + \frac{vv'}{c^2}}$$

Let us define

$$\gamma'' = \frac{1}{\sqrt{1 - \frac{v''^2}{c^2}}}$$

Now,

$$1 - \frac{v''^2}{c^2} = 1 - \frac{1}{c^2}\left(\frac{v + v'}{1 + \frac{vv'}{c^2}}\right)^2 = 1 - \frac{1}{c^2}\frac{(v + v')^2}{\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 + \frac{v^2v'^2}{c^4} + \frac{2vv'}{c^2}\right) - (v^2 + v'^2 + 2vv')}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 + \frac{v^2v'^2}{c^2} + 2vv' - v^2 - v'^2 + 2vv'}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 + \frac{v^2 v'^2}{c^2} - v^2 - v'^2}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 - v^2 - v'^2 + \frac{v^2 v'^2}{c^2}}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 - \frac{v^2}{c^2}\right) - v'^2\left(1 - \frac{v^2}{c^2}\right)}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$\therefore \gamma'' = \frac{1}{\sqrt{1 - \frac{v''^2}{c^2}}} = \frac{1}{\sqrt{\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{\left(1 + \frac{vv'}{c^2}\right)^2}}}$$

$$= \frac{\left(1 + \frac{vv'}{c^2}\right)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}} = \gamma\gamma'\left(1 + \frac{vv'}{c^2}\right) \tag{5}$$

Using equation (5) in equations (3) and (4) we get,

$$x'' = \gamma''(x - v''t) \qquad t'' = \gamma''\left(t - \frac{xv''}{c^2}\right)$$

Also,

$$y'' = y', z'' = z'$$

Thus we see that we can get a direct Lorentz transformation from S to S'' system.

Putting, $v' = 0$ in equation (2) we get,

$$x'' = x', y'' = y', z'' = z', t'' = t'$$

which gives an identical Lorentz transformation from S'' to S' system. This is analogous to the *existence of identity* in a group.

Furthur setting $v' = -v$ we get $v'' = 0$. This shows that corresponding to every Lorentz transformation there *exists an inverse* Lorentz transformation.

∴ All the properties of a group are satisfied by Lorentz transformations and hence it is said to form a *group*.

Q. Show that four dimensional volume element $dxdydzdt$ is invariant under Lorentz transformations.

Q. Show that

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

is invariant under Lorentz transformations.

Thank You