1. What do you mean by space curve? Give one example of space curve.

Sol: A space curve is a curve that lies in three-dimensional space and whose points are represented by three coordinates (x, y, z) depending on a parameter.

Mathematically, a space curve is represented as

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

Example: A helix is a space curve given by

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k},$$

where a and b are constants.

2. Define the osculating plane of a space curve and write its equation in Cartesian coordinates.

Sol: The osculating plane of a space curve at a point is the plane containing the tangent vector and principal normal vector of the curve at that point.

Equivalently, it is the plane perpendicular to the binormal vector.

If the point on the curve is (x_0, y_0, z_0) and the binormal vector is (a, b, c), then the Cartesian equation of the osculating plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

3. Write down the Serret–Frenet formulae for a space curve and define each term involved.

Sol: The Serret–Frenet formulae for a space curve are

$$\frac{d\vec{t}}{ds} = \kappa\vec{n},$$

$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} + \tau\vec{b},$$

$$\frac{d\vec{b}}{ds} = -\tau\vec{n},$$

where

- $\vec{t}$ = unit tangent vector,
- $\vec{n}$ = unit principal normal vector,
- $\vec{b}$ = unit binormal vector,
- s = arc length parameter,
- κ = curvature of the curve,
- τ = torsion of the curve.

4. State the Fundamental Existence Theorem for space curves.

Sol: If $\kappa(s)$ and $\tau(s)$ are continuous functions of a real variable s $(s \geq 0)$, then there exists a space curve for which κ is the curvature, τ is the torsion, and s is the arc length measured from some suitable base point.

5. Write down the first and second fundamental forms of a surface and define their coefficients.

Sol: For a surface parameterized by

$$\vec{r} = \vec{r}(u, v),$$

the first fundamental form is

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where

$$E = \vec{r}_u \cdot \vec{r}_u, \qquad F = \vec{r}_u \cdot \vec{r}_v, \qquad G = \vec{r}_v \cdot \vec{r}_v.$$

The second fundamental form is

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where

$$L = \vec{r}_{uu} \cdot \hat{n}, \qquad M = \vec{r}_{uv} \cdot \hat{n}, \qquad N = \vec{r}_{vv} \cdot \hat{n},$$

and $\hat{n}$ is the unit normal vector to the surface.

6. What is the geometrical meaning of a developable surface?

Sol: A developable surface is a surface that can be unfolded or developed onto a plane without stretching, tearing, or distortion. Geometrically, a developable surface has zero Gaussian curvature at every point.

Examples include the cylinder, cone, and tangent surface of a space curve.

7. Define Gaussian curvature of a surface.

Sol: The Gaussian curvature of a surface is defined as the product of the two principal curvatures:

$$K = k_1 k_2.$$

It measures the intrinsic curvature of the surface at a point.

8. What is a minimal surface?

Sol: A minimal surface is a surface whose mean curvature is zero at every point. Such surfaces locally minimize surface area.

9. Define geodesic on a surface.

Sol: A geodesic is the shortest path between two nearby points on a surface. Equivalently, it is a curve on the surface whose geodesic curvature is zero.

10. What is meant by principal curvatures of a surface?

Sol: Principal curvatures are the maximum and minimum normal curvatures at a point on a surface.

They are denoted by

$$k_1 \quad \text{and} \quad k_2.$$

Developable Surfaces and Minimal Surfaces

A **developable surface** is a ruled surface that can be unfolded onto a plane without stretching or tearing. Equivalently, a developable surface has zero Gaussian curvature at every regular point. Such surfaces are important in geometry, engineering, and manufacturing because sheets of paper or metal can be bent into developable surfaces without distortion.

A ruled surface is generated by a one-parameter family of straight lines. Its vector equation may be written as

$$\vec{r}(u,v) = \vec{a}(u) + v\vec{b}(u),$$

where $\vec{a}(u)$ is the base curve and $\vec{b}(u)$ is the direction vector of the generator.

The tangent vectors are

$$\vec{r}_u = \vec{a}'(u) + v\vec{b}'(u), \qquad \vec{r}_v = \vec{b}(u).$$

The normal vector to the surface is

$$\vec{N} = \vec{r}_u \times \vec{r}_v = (\vec{a}' + v\vec{b}') \times \vec{b}.$$

For a ruled surface to be developable, the tangent plane must remain constant along each generator. This happens when the vectors

$$\vec{a}', \quad \vec{b}, \quad \vec{b}'$$

are linearly dependent. Hence the condition for developability is

$$[\vec{a}', \vec{b}, \vec{b}'] = 0,$$

where $[\vec{a}', \vec{b}, \vec{b}']$ denotes the scalar triple product.

Thus, a ruled surface is developable if and only if

$$(\vec{a}' \times \vec{b}) \cdot \vec{b}' = 0.$$

Examples of developable surfaces:

1. Plane
2. Cylinder
3. Cone
4. Tangent surface of a space curve

Developables associated with space curves:

Let $\vec{r}(s)$ be a space curve parameterized by arc length s. Different developable surfaces can be generated using the Frenet frame vectors.

1. **Tangent developable:**

   $$\vec{R}(s,v) = \vec{r}(s) + v\vec{T}(s),$$

   where $\vec{T}(s)$ is the unit tangent vector. This surface is formed by tangent lines to the curve.

2. **Normal developable:**

   $$\vec{R}(s,v) = \vec{r}(s) + v\vec{N}(s),$$

   where $\vec{N}(s)$ is the principal normal vector.

3. **Binormal developable:**

   $$\vec{R}(s,v) = \vec{r}(s) + v\vec{B}(s),$$

   where $\vec{B}(s)$ is the binormal vector.

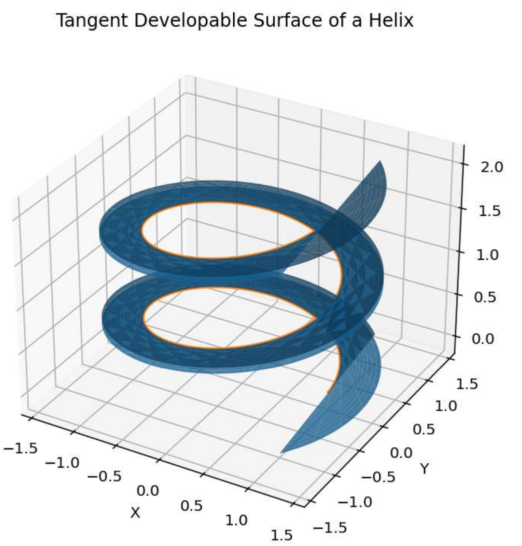

Figure 1: Tangent developable surface of a helix

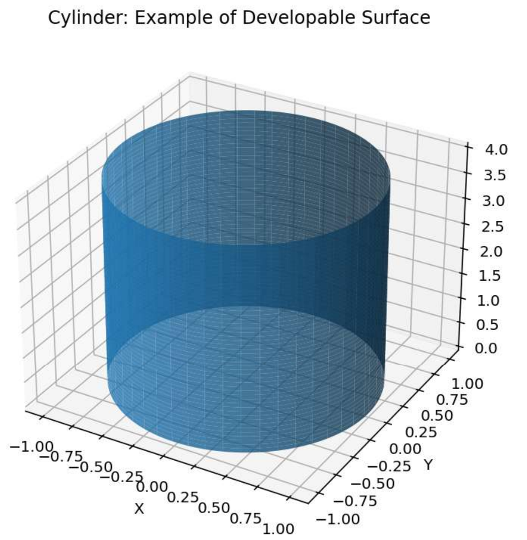

Figure 2: Cylinder as an example of a developable surface

Among these, the tangent developable is always developable because consecutive tangent lines intersect along the curve.

Developables associated with curves on surfaces:

If a curve lies on a surface, then the tangent planes along the curve generate a developable surface called the **envelope of tangent planes**. The geodesic properties and curvature of the curve influence the nature of the developable obtained.

Minimal Surfaces:

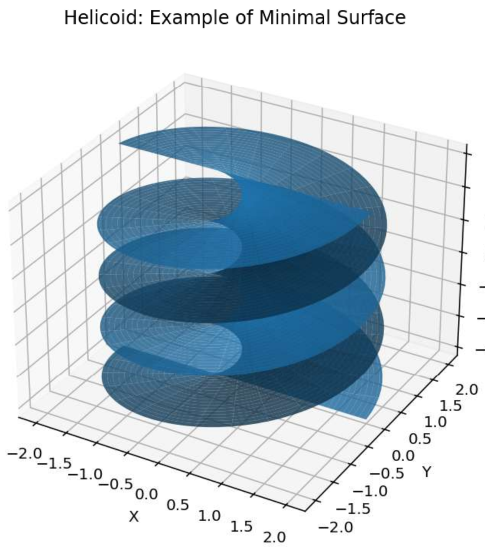

Figure 3: Helicoid as an example of a minimal surface

A **minimal surface** is a surface whose mean curvature is zero at every point, i.e.,

$$H = 0.$$

These surfaces locally minimize area and arise naturally in physical problems such as soap films. If

$$k_1,\ k_2$$

are the principal curvatures, then

$$H = \frac{k_1 + k_2}{2}.$$

Hence for a minimal surface,

$$k_1 + k_2 = 0.$$

Examples of minimal surfaces include:

1. Plane
2. Catenoid
3. Helicoid

Thus, developable surfaces are ruled surfaces with zero Gaussian curvature, while minimal surfaces are characterized by zero mean curvature. Both play an important role in differential geometry and its applications.