

1. **What is the continuum hypothesis? Distinguish between body force and surface force.**

The continuum hypothesis assumes that matter is continuously distributed throughout the body, so that physical quantities such as density, velocity, and stress are defined at every point and vary continuously.

Body force: A force that acts throughout the volume of a body, e.g., gravitational force.

Surface force: A force that acts only on the surface of a body, e.g., pressure and contact forces.

2. **Define infinitesimal strain tensor. What is the physical interpretation of linear strain tensor?**

The infinitesimal strain tensor is defined as

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where u_i are the displacement components.

The linear strain tensor measures small deformations of a body. Its diagonal components represent normal strains (extension or compression), while the off-diagonal components represent shear strains (change in angle).

3. **Given $\sigma_{xx} = 40$, $\sigma_{yy} = 20$, $\sigma_{xy} = 0$, find the principal stresses.**

The principal stresses are given by

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2}.$$

Substituting the values,

$$\begin{aligned} \sigma_{1,2} &= \frac{40 + 20}{2} \pm \sqrt{\left(\frac{40 - 20}{2} \right)^2} \\ &= 30 \pm 10. \end{aligned}$$

Hence,

$$\sigma_1 = 40, \quad \sigma_2 = 20.$$

4. **Why does zero vorticity imply potential flow?**

Vorticity is defined as

$$\vec{\omega} = \nabla \times \vec{V}.$$

If vorticity is zero, then

$$\nabla \times \vec{V} = 0,$$

which means the velocity field is irrotational. Therefore, a scalar potential function ϕ exists such that

$$\vec{V} = \nabla \phi.$$

Hence, the flow is called a potential flow.

5. **State the angular momentum principle. What is the implication of angular momentum on stress tensor?**

The angular momentum principle states that the total torque acting on a continuum equals the rate of change of angular momentum.

For a continuum without body couples, this principle implies that the stress tensor is symmetric:

$$\sigma_{ij} = \sigma_{ji}.$$

6. What is the physical meaning of equilibrium in a continuum?

Equilibrium in a continuum means that the resultant force and resultant moment acting on every part of the body are zero. Therefore, the body remains at rest or moves with constant velocity without acceleration.

7. Define material derivative and local derivative.

The local derivative represents the rate of change of a fluid property with respect to time at a fixed point in space:

$$\frac{\partial \phi}{\partial t}.$$

The material derivative represents the total rate of change experienced by a moving fluid particle:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + (\vec{V} \cdot \nabla)\phi.$$

8. What is meant by irrotational motion? Define potential flow.

A flow is called irrotational if its vorticity is zero:

$$\nabla \times \vec{V} = 0.$$

In such a case, the velocity vector can be expressed as the gradient of a scalar potential function:

$$\vec{V} = \nabla \phi.$$

This type of flow is called potential flow.

9. State Reynolds transport theorem.

Reynolds transport theorem relates the rate of change of an extensive property of a system to the properties within a control volume. It is given by

$$\frac{d}{dt} \int_{\text{system}} B = \frac{\partial}{\partial t} \int_{CV} \beta \rho dV + \int_{CS} \beta \rho (\vec{V} \cdot \vec{n}) dA,$$

where $\beta = B/m$.

10. Distinguish between viscous and inviscid fluids.

A viscous fluid possesses internal friction due to viscosity and resists relative motion between fluid layers.

An inviscid fluid is an ideal fluid with zero viscosity and hence no internal friction.

11. What is the continuity equation?

The continuity equation represents the conservation of mass in fluid flow. For a compressible fluid it is written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0.$$

For an incompressible fluid,

$$\nabla \cdot \vec{V} = 0.$$

Q: Starting from the conservation of mass, obtain the differential form of the continuity equation.

Solution: The continuity equation is obtained from the fundamental law of conservation of mass. The law states that:

Mass can neither be created nor destroyed.

Therefore, for any fluid motion, the total mass of fluid must remain conserved.

Let us consider a fixed control volume V enclosed by a closed surface S .

Let,

- ρ = density of the fluid,
- \vec{V} = velocity vector of the fluid,
- \vec{n} = outward unit normal vector to the surface.

At any instant, the total mass inside the control volume is

$$\int_V \rho dV.$$

According to conservation of mass,

Rate of increase of mass inside the control volume

+

Net outward flow of mass through the surface

= 0.

Mathematically,

$$\frac{d}{dt} \int_V \rho dV + \oint_S \rho \vec{V} \cdot \vec{n} dS = 0.$$

1.

$$\frac{d}{dt} \int_V \rho dV$$

represents the rate of change of mass inside the control volume.

2.

$$\oint_S \rho \vec{V} \cdot \vec{n} dS$$

represents the net mass flux leaving the control surface.

Using the divergence theorem,

$$\oint_S \rho \vec{V} \cdot \vec{n} dS = \int_V \nabla \cdot (\rho \vec{V}) dV.$$

Substituting into the conservation equation, we get

$$\frac{d}{dt} \int_V \rho dV + \int_V \nabla \cdot (\rho \vec{V}) dV = 0.$$

Since the control volume is fixed in space, the time derivative can be taken inside the integral:

$$\frac{d}{dt} \int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV.$$

Therefore,

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot (\rho \vec{V}) dV = 0.$$

Combining the two volume integrals,

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] dV = 0.$$

The above equation is valid for any arbitrary control volume V .

Hence, the integrand itself must vanish everywhere inside the fluid. Therefore,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$

This is the **differential form of the continuity equation** for a compressible fluid.

Continuity Equation for Incompressible Flow

For an incompressible fluid, density remains constant, i.e.,

$$\rho = \text{constant}.$$

Hence,

$$\nabla \cdot \vec{V} = 0.$$

Therefore, the continuity equation for incompressible flow becomes

$$\boxed{\nabla \cdot \vec{V} = 0}$$

Thus, starting from the conservation of mass principle, we obtained the differential form of the continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$

which governs the conservation of mass in fluid mechanics and continuum mechanics.

Q: Explain the concept of fluid pressure and show why it acts normal to a surface.

Solution: Fluid pressure is defined as the normal force exerted by a fluid per unit area on any surface in contact with it.

Mathematically,

$$p = \frac{F}{A},$$

where

- p = fluid pressure,

- F = normal force exerted by the fluid,
- A = area of the surface.

Pressure is a scalar quantity and its SI unit is Pascal (Pa).

In a fluid at rest, molecules move randomly and continuously collide with one another and with the walls of the container. These molecular collisions produce a force on the surface, giving rise to pressure.

Let us consider a small surface element inside a fluid at rest and the fluid exerts a force on this surface. This force may be resolved into

- a normal component,
- a tangential component.

If a tangential component of force existed, it would produce shear stress in the fluid.

However, a fluid at rest cannot sustain shear stress. Even a very small shear stress would cause the fluid layers to move relative to each other, and the fluid would no longer remain at rest.

Therefore, for a fluid in equilibrium,

$$\text{Shear stress} = 0.$$

Hence, only the normal component of force can exist.

Thus, the pressure force always acts perpendicular (normal) to the surface.

Mathematical Explanation

Let the force acting on a small area element dA be $d\vec{F}$.

The pressure is defined as

$$p = \lim_{dA \rightarrow 0} \frac{dF_n}{dA},$$

where dF_n is the normal component of the force.

Since no tangential component exists in a fluid at rest,

$$d\vec{F} = -p dA \vec{n},$$

where

- \vec{n} is the outward unit normal vector,
- the negative sign indicates that pressure acts inward on the surface.

Thus, the force due to pressure is always normal to the surface.

Hence, fluid pressure is the normal force per unit area exerted by a fluid, and it acts normal to a surface because a fluid at rest cannot sustain shear stress.

Q: State the angular momentum principle and show that it leads to the symmetry of the stress tensor.

Solution: Angular Momentum Principle:

The angular momentum principle states that:

The resultant moment acting on a continuum body about any point is equal to the rate of change of angular momentum of the body about that point.

For a continuum in equilibrium and in the absence of body couples, the total moment acting on an element must vanish.

Let us consider a small rectangular element of dimensions

$$dx, \quad dy, \quad dz$$

in a continuum.

Let the stress components acting on the faces be:

$$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$$

for normal stresses, and

$$\sigma_{xy}, \sigma_{yx}, \sigma_{yz}, \sigma_{zy}, \sigma_{zx}, \sigma_{xz}$$

for shear stresses.

To prove symmetry, consider moments about the z-axis.

The shear stress σ_{xy} acts on the face normal to the x-direction, while σ_{yx} acts on the face normal to the y-direction.

Moment Due to Shear Stress σ_{xy}

Force due to σ_{xy} :

$$\sigma_{xy}(dy \, dz)$$

Moment arm about the center:

$$\frac{dx}{2}$$

Hence, the moment is

$$\sigma_{xy}(dy \, dz) \left(\frac{dx}{2} \right).$$

Considering both opposite faces,

$$M_1 = \sigma_{xy} dx \, dy \, dz.$$

Moment Due to Shear Stress σ_{yx}

Similarly,

$$M_2 = \sigma_{yx} dx \, dy \, dz.$$

This moment acts in the opposite direction.

For rotational equilibrium,

$$M_1 - M_2 = 0.$$

Therefore,

$$\sigma_{xy} dx dy dz - \sigma_{yx} dx dy dz = 0.$$

Dividing by the nonzero volume element $dx dy dz$,

$$\sigma_{xy} = \sigma_{yx}.$$

Similarly, considering moments about the other coordinate axes, we obtain

$$\sigma_{yz} = \sigma_{zy},$$

and

$$\sigma_{zx} = \sigma_{xz}.$$

Hence, the stress tensor satisfies

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

Thus, the angular momentum principle implies that the stress tensor is symmetric in the absence of body couples. This symmetry reduces the number of independent stress components from nine to six.

Q: Describe in detail the Lagrangian and Eulerian descriptions of motion. Compare both approaches with suitable examples and discuss their applications in continuum mechanics.

Solution: In continuum mechanics, the motion of a material body or fluid can be described in two different ways:

1. Lagrangian description
2. Eulerian description

These descriptions provide mathematical methods to study deformation, velocity, acceleration, stress, and strain in solids and fluids.

The main difference between the two approaches lies in the way the observer studies the motion.

- In the **Lagrangian approach**, the observer follows individual material particles as they move.
- In the **Eulerian approach**, the observer studies what happens at fixed points in space.

Both descriptions are fundamental in continuum mechanics, fluid mechanics, elasticity, and computational mechanics.

Lagrangian Description of Motion

In the Lagrangian description, attention is focused on individual particles of the continuum.

Each particle is identified by its initial position at time

$$t = 0.$$

The observer follows the motion of the same particle as time progresses.

Thus, the history of every particle is tracked throughout the motion.

Material Coordinates

Suppose a particle initially occupies the position

$$(X, Y, Z)$$

in the reference configuration.

These coordinates are called:

- material coordinates,
- reference coordinates,
- Lagrangian coordinates.

At time t , the particle moves to the new position

$$(x, y, z).$$

Hence, the motion is described by

$$x = x(X, Y, Z, t),$$

$$y = y(X, Y, Z, t),$$

$$z = z(X, Y, Z, t).$$

Or in vector form,

$$\vec{x} = \vec{x}(\vec{X}, t).$$

Velocity in Lagrangian Description

The velocity of the particle is obtained by differentiating the position vector with respect to time while keeping the material coordinates fixed:

$$\vec{v} = \frac{\partial \vec{x}}{\partial t}.$$

Acceleration in Lagrangian Description

Similarly, acceleration is

$$\vec{a} = \frac{\partial^2 \vec{x}}{\partial t^2}.$$

Physical Interpretation

The Lagrangian observer moves together with the particle.

It is similar to attaching a camera to a moving particle and observing its entire path.

Example of Lagrangian Description

Suppose a fluid particle initially at

$$(X, Y, Z) = (1, 2, 0)$$

moves according to

$$x = X + t,$$

$$y = Y + 2t,$$

$$z = Z.$$

Then, at any time t , the exact position of that same particle can be determined.

Thus, the trajectory of the particle is completely known.

Advantages of Lagrangian Description

- Gives complete history of each particle.
- Convenient for studying deformation and strain.
- Useful in solid mechanics and elasticity.
- Easy to track boundaries and interfaces.

Disadvantages of Lagrangian Description

- Difficult for large fluid motions.
- Tracking millions of particles becomes complicated.
- Mathematical equations become complex for turbulent flows.

Eulerian Description of Motion

In the Eulerian description, attention is focused on fixed points in space rather than individual particles.

The observer studies how fluid properties change at a fixed spatial location as different particles pass through that point.

Spatial Coordinates

The position in space is specified by coordinates

$$(x, y, z).$$

Physical quantities such as velocity, pressure, density, and temperature are expressed as functions of space and time:

$$\vec{V} = \vec{V}(x, y, z, t),$$

$$p = p(x, y, z, t),$$

$$\rho = \rho(x, y, z, t).$$

Velocity Field

In the Eulerian description, velocity is treated as a field variable.

For example,

$$\vec{V}(x, y, z, t) = u(x, y, z, t)\hat{i} + v(x, y, z, t)\hat{j} + w(x, y, z, t)\hat{k}.$$

This gives the velocity of whichever particle passes through the point (x, y, z) at time t .

Physical Interpretation

The observer remains fixed in space.

This is similar to placing a camera at a fixed location and observing particles flowing past it.

Example of Eulerian Description

Suppose the velocity field is

$$\vec{V} = 2x\hat{i} + 3y\hat{j}.$$

At the point

$$(1, 1),$$

the velocity is

$$\vec{V} = 2\hat{i} + 3\hat{j}.$$

This describes the velocity at that location, irrespective of which particle is present there.

Advantages of Eulerian Description

- Suitable for fluid flow problems.
- Easier for studying steady and unsteady flows.
- Convenient for practical measurements.
- Widely used in fluid dynamics and CFD.

Disadvantages of Eulerian Description

- Does not directly track individual particles.
- Deformation history is not immediately available.
- Material properties require additional calculations.

Comparison Between Lagrangian and Eulerian Descriptions

Feature	Lagrangian Description	Eulerian Description
Reference frame	Follows moving particles	Fixed in space
Coordinates used	Material coordinates (X, Y, Z)	Spatial coordinates (x, y, z)
Observer motion	Moves with particle	Remains stationary
Main focus	Individual particles	Field variables at fixed points
Best suited for	Solid mechanics	Fluid mechanics
Velocity representation	Function of particle and time	Function of space and time
Difficulty level	Difficult for large flows	Easier for fluid flow analysis
Examples	Elastic deformation of solids	Airflow, water flow

Applications of Lagrangian Description

- Solid mechanics
- Elasticity and plasticity
- Structural deformation
- Finite element analysis
- Tracking crack propagation

Applications of Eulerian Description

- Fluid mechanics
- Aerodynamics
- Hydrodynamics
- Weather prediction
- Computational fluid dynamics (CFD)

Relation Between the Two Descriptions

Both descriptions are mathematically equivalent and can be transformed into one another. The material derivative connects the Eulerian and Lagrangian viewpoints:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla).$$

Here,

$$\frac{D}{Dt}$$

represents the rate of change following a moving particle.

The Lagrangian and Eulerian descriptions are two fundamental approaches for describing motion in continuum mechanics.

The Lagrangian approach follows individual particles and is mainly useful in solid mechanics and deformation analysis.

The Eulerian approach studies field variables at fixed points in space and is widely used in fluid mechanics and flow analysis.

Both methods are essential for understanding the mechanics of continuous media and form the foundation of modern continuum mechanics and computational modeling.