UNIT 3 - Fluid Dynamics II

Presented by

MATHEMATICAL EXPLORATIONS

Explore Mathematical Cosmos

Dynamical Similarity

Dynamical similarity is a fundamental concept in fluid mechanics and model analysis, used to ensure that a model and its prototype behave in a physically similar way under scaled conditions.

It is especially important in experiments such as wind tunnel testing, hydraulic models, and aerodynamics.

Definition

Two systems (model and prototype) are said to be dynamically similar if the ratios of all corresponding forces acting on them are equal. This means that the motion of fluid particles in both systems follows the same pattern.

Conditions for Similarity

For complete similarity, three conditions must be satisfied:

(i) Geometric Similarity

- Model and prototype have the same shape.
- Ratio of corresponding lengths is constant.

$$\frac{L_m}{L_p} = \text{constant}$$

(ii) Kinematic Similarity

- Motion of fluid is similar.
- Velocity ratios at corresponding points are equal.

$$\frac{V_m}{V_p} = \text{constant}$$

(iii) Dynamical Similarity

- Ratios of forces are equal.
- Governs actual physical behavior.

Force Ratios and Dimensionless Numbers

Dynamical similarity between a model and its prototype is achieved when the ratios of all significant forces acting in the two systems are equal. Instead of comparing individual forces directly, this condition is conveniently expressed in terms of dimensionless numbers. These non-dimensional parameters represent the ratio of different types of forces and must be the same for both the model and the prototype.

(a) Reynolds Number (Viscous Force)

The Reynolds number is a fundamental dimensionless parameter in fluid mechanics that represents the ratio of inertial forces to viscous forces in a fluid flow. It is widely used to characterize the nature of the flow.

Reynold's number is given by

$$Re = \frac{\rho V L}{\mu}$$

where,

- ρ = density of the fluid,
- V = characteristic velocity,
- L = characteristic length,
- μ = dynamic viscosity of the fluid.

It can also be expressed as:

$$Re = \frac{VL}{\nu}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.

Physical Significance

The Reynolds number indicates the relative importance of inertial and viscous forces:

- Low Reynolds number ($Re \ll 1$): Viscous forces dominate, resulting in smooth and orderly (laminar) flow.
- High Reynolds number ($Re \gg 1$): Inertial forces dominate, leading to irregular and chaotic (turbulent) flow.

Flow Regimes in Pipe Flow

- Laminar flow: $Re < 2000$
- Transitional flow: $2000 < Re < 4000$
- Turbulent flow: $Re > 4000$

Importance in Dynamical Similarity

The Reynolds number plays a crucial role in achieving dynamical similarity between a model and its prototype.

$$Re_m = Re_p$$

Ensuring equality of Reynolds number guarantees that the ratio of inertial to viscous forces is the same in both systems, thereby maintaining similarity in flow behavior.

Applications

- Flow through pipes and channels
- Boundary layer analysis

- Aerodynamics and hydrodynamics
- Design of hydraulic and mechanical systems

(b) Froude Number

The Froude number is an important dimensionless parameter in fluid mechanics that represents the ratio of inertial forces to gravitational forces. It is particularly significant in flows where the free surface plays a dominant role.

Definition

$$Fr = \frac{V}{\sqrt{gL}}$$

where,

- V = characteristic velocity of the flow,
- g = acceleration due to gravity,
- L = characteristic length (such as depth of flow).

Froude Number Physical Significance

The Froude number indicates the relative influence of inertia and gravity on fluid motion:

- Low Froude number ($Fr < 1$): Gravitational forces dominate; flow is subcritical (slow and stable).
- $Fr = 1$: Critical flow condition.
- High Froude number ($Fr > 1$): Inertial forces dominate; flow is supercritical (rapid and unstable).

Types of Flow Based on Froude Number

- Subcritical flow: $Fr < 1$
- Critical flow: $Fr = 1$
- Supercritical flow: $Fr > 1$

Importance in Dynamical Similarity

The Froude number is essential in problems involving free surface flows such as rivers, canals, and wave motion.

$$Fr_m = Fr_p$$

Ensuring equality of the Froude number between model and prototype guarantees that the ratio of inertial to gravitational forces remains the same, thereby preserving the flow characteristics.

- Open channel flow (rivers, canals)
- Hydraulic structures (spillways, dams)
- Ship hydrodynamics
- Wave motion analysis

(c) Mach Number

The Mach number is a fundamental dimensionless parameter in fluid mechanics and compressible flow that represents the ratio of flow velocity to the speed of sound in the medium. It is especially important in high-speed aerodynamics and gas dynamics.

Definition

$$M = \frac{V}{a}$$

where,

- V = velocity of the flow,
- a = speed of sound in the medium.

Mach Number Physical Significance

The Mach number indicates the relative importance of inertial effects compared to compressibility effects in a fluid:

- Low Mach number ($M < 1$): Flow is subsonic; compressibility effects are negligible.
- $M = 1$: Sonic condition; flow velocity equals the speed of sound.
- High Mach number ($M > 1$): Flow is supersonic; compressibility effects are significant.

Types of Flow Based on Mach Number

- Subsonic flow: $M < 1$
- Sonic flow: $M = 1$
- Supersonic flow: $1 < M < 5$
- Hypersonic flow: $M > 5$

Importance in Dynamical Similarity

The Mach number is crucial in compressible flow modeling, particularly in aerodynamics and gas flow systems:

$$M_m = M_p$$

Ensuring equality of the Mach number between model and prototype guarantees that compressibility effects are properly represented, maintaining similarity in pressure, density, and temperature variations.

- Aerodynamics of aircraft and rockets
- Supersonic and hypersonic flow analysis
- Nozzle and jet propulsion systems
- Shock wave and expansion wave studies

(d) Weber Number

The Weber number is an important dimensionless parameter in fluid mechanics that represents the ratio of inertial forces to surface tension forces. It is particularly significant in flows involving interfaces between fluids, such as droplets, bubbles, and liquid jets.

Definition

$$We = \frac{\rho V^2 L}{\sigma}$$

where,

- ρ = density of the fluid,
- V = characteristic velocity of the flow,
- L = characteristic length (such as droplet diameter),
- σ = surface tension of the fluid.

Weber Number Physical Significance

The Weber number indicates the relative influence of inertia and surface tension on fluid motion:

- Low Weber number ($We < 1$): Surface tension forces dominate; droplets tend to remain intact and stable.
- $We \approx 1$: Balance between inertial and surface tension forces.
- High Weber number ($We > 1$): Inertial forces dominate; droplets may deform or break up.

Types of Flow Based on Weber Number

- Surface tension dominated flow: $We < 1$
- Transitional regime: $We \approx 1$
- Inertia dominated flow: $We > 1$

Importance in Dynamical Similarity

The Weber number is essential in problems involving multiphase flows and interfacial phenomena:

$$We_m = We_p$$

Ensuring equality of the Weber number between model and prototype guarantees that the balance between inertial and surface tension forces is preserved, thereby maintaining similar deformation and breakup characteristics.

- Droplet and bubble dynamics
- Atomization and spray formation
- Inkjet printing technology
- Fuel injection systems

Condition for Dynamical Similarity

For complete dynamical similarity between a model and its prototype, all relevant dimensionless numbers must be equal. This ensures that the ratios of corresponding forces are identical in both systems.

$$Re_m = Re_p, \quad Fr_m = Fr_p, \quad Ma_m = Ma_p, \quad We_m = We_p$$

However, in practical situations, it is often difficult to satisfy all similarity conditions simultaneously. Therefore, the most dominant forces in the system are identified, and the corresponding dimensionless numbers are matched.

- In viscous-dominated flows: Reynolds number similarity is maintained.
- In gravity-dominated flows: Froude number similarity is maintained.
- In compressible flows: Mach number similarity is maintained.
- In surface tension-dominated flows: Weber number similarity is maintained.

Thus, achieving dynamical similarity involves careful selection of the governing forces and ensuring equality of the corresponding dimensionless parameters.

Dimensional analysis

The dimensional analysis is a mathematical technique, which enables us to obtain the dimensionless numbers of the variables of a physical problem without the knowledge of the governing equation. This analysis is based on the assumption that each physical phenomenon is expressible in terms of a dimensionally homogeneous equation. It is used in developing experiments with several variables, in interpreting experimental data and in establishing link between the scale model (or simply model) and the actual structure (also known as prototype) after performing suitable experiments.

Technique of dimensional analysis

We now propose to discuss the following two techniques of dimensional analysis:

(i) Rayleigh's technique.

(ii) Buckingham π–theorem.

Rayleigh's technique (working rule)

Step 1. An arbitrary functional relationship is assumed among the variables of the problem. The choice of the dependent and independent variables is made as required by the problem.

Step 2. The equation of step 1 is now re-written by using a constant k and raising the independent variables to powers $a, b, c, \ldots$ etc.

Step 3. The principle of dimensional homogeneity is used to compare the powers of fundamental units on both sides. This leads to some simultaneous equations to determine $a, b, c, \ldots$ etc.

Step 4. The substitution of the values of $a, b, c, \ldots$ in the proposed relation of step 2 leads to the desired result.

Ex. Show that the resistance (R) to the motion of a sphere of diameter (D) moving with a uniform velocity (V) through a real fluid having density ρ and viscosity (μ) is given by

$$R = \rho D^2 V^2 f\left(\frac{\mu}{\rho V D}\right)$$

Sol. Let

$$R = F(D, V, \rho, \mu) \qquad \ldots(1)$$

Let k be a dimensionless constant. Then (1) can be re-written as

$$R = k\left[D^a \cdot V^b \cdot \rho^c \cdot \mu^d\right] \qquad \ldots(2)$$

Quantity	Symbol	Dimensions
Resistance (force)	R	$[MLT^{-2}]$
Diameter	D	$[L]$
Velocity	V	$[LT^{-1}]$
Density	ρ	$[ML^{-3}]$
Viscosity	μ	$[ML^{-1}T^{-1}]$

Table 1: Dimensions of physical quantities

Substituting the dimensions of each physical quantity, (2) reduces to

$$[MLT^{-2}] = k\left[(L)^a(LT^{-1})^b(ML^{-3})^c(ML^{-1}T^{-1})^d\right]$$

or

$$[MLT^{-2}] = k\left[M^{c+d}L^{a+b-3c-d}T^{-b-d}\right] \qquad \ldots(3)$$

Since (3) must be dimensionally homogeneous, we equate the powers of M, L and T and obtain

$$c + d = 1 \qquad \ldots (4)$$

$$a + b - 3c - d = 1 \qquad \ldots (5)$$

$$-b - d = -2 \qquad \ldots (6)$$

From (4) and (6),

$$c = 1 - d \qquad \text{and} \qquad b = 2 - d \qquad \ldots (7)$$

Substituting the values of c and b in (5), we get

$$a = 1 - b + 3c + d = 1 - (2 - d) + 3(1 - d) + d = 2 - d \qquad \ldots (8)$$

Using (7) and (8) in (2), we get

$$R = kD^{2-d}V^{2-d}\rho^{1-d}\mu^{d} = \rho D^2 V^2\, k\left(\frac{\mu}{\rho V D}\right)^{d} \qquad \ldots(9)$$

Since d and k are arbitrary constants, we may take

$$k\left(\frac{\mu}{\rho V D}\right)^{d} = f\left(\frac{\mu}{\rho V D}\right), \qquad \ldots(10)$$

where f is an arbitrary function. Using (10), (9) yields the required result

$$R = \rho D^2 V^2\, f\left(\frac{\mu}{\rho V D}\right).$$

Buckingham π–theorem or simply π–theorem

Statement. If $Q_1, Q_2, \ldots, Q_n$ be n physical quantities involved in a physical phenomenon and if there are m independent fundamental units in this system then a relation

$$\phi(Q_1, Q_2, \ldots, Q_n) = 0$$

is equivalent to the relation

$$f(\pi_1, \pi_2, \ldots, \pi_{n-r}) = 0,$$

where $\pi_1, \pi_2, \ldots, \pi_{n-r}$ are the dimensionless quantities formed by $Q_1, Q_2, \ldots, Q_n$ and r is the rank of the dimensional matrix of the given physical quantities.

An alternative statement of π–theorem

An equation in physical variables which is dimensionally homogeneous, can be reduced to a relationship among a complete set of dimensionless products.

Proof. The π–theorem is based on the following assumptions:

(i) It is always possible to select m independent fundamental units in a physical phenomenon. (For example, in the case of viscous incompressible fluid, $m = 3$, i.e. mass, length and time. Again for the case of viscous compressible fluid, $m = 4$, i.e. mass, length, time and temperature).

(ii) There exists n quantities say $Q_1, Q_2, \ldots, Q_n$ involved in a physical phenomenon whose dimensional formulae may be expressed in terms of m fundamental units.

(iii) There exists a fundamental relationship between n dimensional quantities $Q_1, Q_2, \ldots, Q_n$ say

$$\phi(Q_1, Q_2, \ldots, Q_n) = 0 \qquad \ldots(1)$$

and this equation is independent of the types of units and is dimensionally homogeneous.

Let dimensions of $Q_1, Q_2, \ldots, Q_n$ be expressed in terms of m fundamental units $f_1, f_2, \ldots, f_m$ as follows:

$$Q_1 = f_1^{a_{11}} f_2^{a_{21}} \cdots f_m^{a_{m1}}, \qquad Q_2 = f_1^{a_{12}} f_2^{a_{22}} \cdots f_m^{a_{m2}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Q_n = f_1^{a_{1n}} f_2^{a_{2n}} \cdots f_m^{a_{mn}}$$

and

The matrix of dimensions (i.e., dimensional matrix) of the given physical n quantities $Q_1, Q_2, \ldots, Q_n$ may be expressed as follows:

$$\begin{array}{c|cccc} & Q_1 & Q_2 & \cdots & Q_n \\ \hline f_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ f_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \qquad \ldots(3)$$

Let the above $m \times n$ matrix be denoted by A.
Let us form a dimensionless product of powers of $Q_1, Q_2, \ldots, Q_n$ as follows:

$$\pi = Q_1^{x_1} Q_2^{x_2} \cdots Q_n^{x_n} \qquad \ldots(4)$$

Since product π is dimensionless, we have

$$\pi = f_1^0 f_2^0 \cdots f_m^0 \qquad \ldots(5)$$

Substituting the dimensions of π, $Q_1, Q_2, \ldots, Q_n$ from (5) and (2) in (4), we get

$$f_1^0 f_2^0 \cdots f_m^0 = (f_1^{a_{11}} f_2^{a_{21}} \cdots f_m^{a_{m1}})^{x_1} (f_1^{a_{12}} f_2^{a_{22}} \cdots f_m^{a_{m2}})^{x_2} \cdots (f_1^{a_{1n}} f_2^{a_{2n}} \cdots f_m^{a_{mn}})^{x_n}$$

Since the above equation must be dimensionally homogeneous, the exponents of each dimension on both sides of (2) must be identical. Thus, we have

$$\begin{aligned} \text{For } f_1 : &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \text{For } f_2 : &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ &\quad \cdots\cdots\cdots\cdots\cdots \\ \text{For } f_m : &\quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{aligned} \qquad \ldots(6)$$

Rewriting (6) in matrix form, we have

$$AX = 0 \qquad \ldots(7)$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots (8)$$

Now, (6) is a set of m homogeneous equations in n unknowns. Let r be the rank of the dimensional matrix A. Then we conclude that the number of linearly independent solutions of the equations are $n-r$. It follows that corresponding to each independent solution of X we will have a dimensionless product π and hence the number of dimensionless products in a complete set will be $n - r$.

Ex. The pressure difference Δp in a pipe of diameter D and length l due to turbulent flow depends on the velocity V, viscosity μ, density ρ and roughness k. Using Buckingham's theorem, obtain an expression for Δp.

Sol. Here

$$\Delta p = f(D, l, V, \mu, \rho, k) \quad \text{or} \quad f(\Delta p, D, l, V, \mu, \rho, k) = 0 \qquad \ldots (1)$$

where f and f_1 are arbitrary functions.
Hence n = total number of variables = 7.
Here, writing dimensions of each variable, we have

$$\Delta p = ML^{-1}T^{-2}, \quad D = L, \quad l = L, \quad V = LT^{-1},$$

$$\mu = ML^{-1}T^{-1}, \quad \rho = ML^{-3}, \quad k = L.$$

Hence, m = total number of fundamental units = 3.

So according to Buckingham's π–theorem, the number of dimensionless numbers $= n - m = 7 - 3 = 4$.
Let π_1, π_2, π_3 and π_4 be dimensionless numbers. Then (1) can be rewritten as

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \qquad \ldots (2)$$

We select D, V and ρ as $(= 3)$ repeating variables.
We now form each dimensionless π–number by combining repeating variables D, V and ρ with one of the remaining variables $\Delta p, l, \mu, k$. Note that each π–number contains $m + 1 (= 3 + 1 = 4)$ variables.

$$\pi_1 = D^{a_1} V^{b_1} \rho^{c_1} \cdot \Delta p, \qquad \ldots (3)$$

$$\pi_2 = D^{a_2} V^{b_2} \rho^{c_2} \cdot l, \qquad \ldots (4)$$

$$\pi_3 = D^{a_3} V^{b_3} \rho^{c_3} \cdot \mu, \qquad \ldots (5)$$

$$\pi_4 = D^{a_4} V^{b_4} \rho^{c_4} \cdot k, \qquad \ldots (6)$$

where $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ are constants.

Determination of π_1: Substituting the dimensions of each physical quantity in (3) and noting that π_1 is a dimensionless number, we get

$$M^0 L^0 T^0 = L^{a_1} (LT^{-1})^{b_1} (ML^{-3})^{c_1} (ML^{-1}T^{-2}) \qquad \ldots (7)$$

Since (7) must be dimensionally homogeneous, equate the exponents of M, L, T on both sides and obtain

$$0 = c_1 + 1, \qquad 0 = a_1 + b_1 - 3c_1 - 1, \qquad 0 = -b_1 - 2$$

Solving these,

$$a_1 = 0, \qquad b_1 = -2, \qquad c_1 = -1$$

$$\therefore \quad \pi_1 = D^0 V^{-2} \rho^{-1} \cdot \Delta p = \frac{\Delta p}{\rho V^2} \qquad \ldots (8)$$

Determination of π_2. As before (4) gives

$$M^0 L^0 T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot L. \tag{9}$$

$$(9) \Rightarrow \quad 0 = c_2, \quad 0 = a_2 + b_2 - 3c_2 + 1 \quad \text{and} \quad 0 = -b_2.$$

Solving these,

$$a_2 = -1, \quad b_2 = 0, \quad c_2 = 0.$$

$$\therefore (4) \text{ gives}, \quad \pi_2 = D^{-1} V^0 \rho^0 \cdot l = \frac{l}{D}. \tag{10}$$

Determination of π_3. As before (5) gives

$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot (ML^{-1}T^{-1}). \tag{11}$$

$$(11) \Rightarrow \quad 0 = c_3 + 1, \quad 0 = a_3 + b_3 - 3c_3 - 1 \quad \text{and} \quad 0 = -b_3 - 1.$$

Solving these,

$$a_3 = -1, \quad b_3 = -1, \quad c_3 = -1.$$

$$(5) \text{ gives}, \quad \pi_3 = D^{-1}V^{-1}\rho^{-1} \cdot \mu = \frac{\mu}{DV\rho}. \tag{12}$$

Determination of π_4. As before (6) gives

$$M^0L^0T^0 = L^{a_4} \cdot (LT^{-1})^{b_4} \cdot (ML^{-3})^{c_4} \cdot L. \tag{13}$$

$$(13) \Rightarrow \quad 0 = c_4, \quad 0 = a_4 + b_4 - 3c_4 + 1 \quad \text{and} \quad 0 = -b_4.$$

Solving these,

$$a_4 = -1, \quad b_4 = 0, \quad c_4 = 0.$$

$$(6) \text{ gives}, \quad \pi_4 = D^{-1} V^0 \rho^0 \cdot k = \frac{k}{D}. \tag{14}$$

Using (8), (10), (12) and (14), (2) reduces to

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right) = 0 \quad \text{or} \quad \frac{\Delta p}{\rho V^2} = \phi\left(\frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right).$$

So that

$$\Delta p = \rho V^2 \phi\left(\frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right).$$

Magnetic Reynolds Number

The magnetic Reynolds number is an important dimensionless parameter in magnetohydrodynamics (MHD) that represents the ratio of magnetic field advection by the fluid motion to magnetic diffusion due to finite electrical conductivity.

Definition

$$R_m = \frac{\mu\sigma VL}{1} = \mu\sigma VL$$

or equivalently,

$$R_m = \frac{VL}{\eta}$$

where,

- V = characteristic velocity of the fluid,
- L = characteristic length scale,

- μ = magnetic permeability of the medium,
- σ = electrical conductivity,
- $\eta = \frac{1}{\mu\sigma}$ = magnetic diffusivity.

Magnetic Reynolds Number Physical Significance

The magnetic Reynolds number indicates the relative importance of magnetic advection to magnetic diffusion:

- Low magnetic Reynolds number ($R_m \ll 1$): Magnetic diffusion dominates; magnetic field lines are not carried with the fluid.
- $R_m \approx 1$: Both advection and diffusion effects are comparable.
- High magnetic Reynolds number ($R_m \gg 1$): Advection dominates; magnetic field lines are "frozen" into the fluid (flux freezing condition).

Types of Flow Based on Magnetic Reynolds Number

- Diffusion-dominated regime: $R_m \ll 1$
- Transitional regime: $R_m \approx 1$

- Advection-dominated (frozen-in field): $R_m \gg 1$

Importance in Dynamical Similarity

The magnetic Reynolds number is crucial in MHD modeling and astrophysical flows:

$$(R_m)_m = (R_m)_p$$

Ensuring equality of magnetic Reynolds number between model and prototype ensures that the balance between magnetic advection and diffusion is preserved.

- Astrophysical plasmas (stars, galaxies, accretion disks)
- Geodynamo and planetary magnetic fields
- Fusion plasma confinement (tokamaks)
- Liquid metal flows and MHD generators