Intrinsic Properties of a Surface
Differential Geometry

Presented by

MATHEMATICAL EXPLORATIONS

Explore Mathematical Cosmos

Important Relation Between the Coefficients E, F, G and H:

From vector identity

$$(\vec{r}_1 \times \vec{r}_2)^2 = r_1^2 r_2^2 - (\vec{r}_1 \cdot \vec{r}_2)^2$$

We have

$$H^2 = EG - F^2, \quad \text{where } H = |\vec{r}_1 \times \vec{r}_2|$$

Since H is a positive quantity, so $EG - F^2$ is always positive and

$$H = \sqrt{EG - F^2}$$

Again, at an ordinary point $r_1 \neq 0, r_2 \neq 0$, thus

$$E = r_1^2 > 0, \quad G = r_2^2 > 0$$

Hence we have

$$E > 0, \quad G > 0, \quad EG - F^2 > 0$$

Important Properties of the Metric:

Property 1. The metric or the first fundamental form is a positive definite quadratic form in du, dv.

Proof. Since $E > 0$,

$$\begin{aligned} E\,du^2 + 2F\,du\,dv + G\,dv^2 &= \frac{1}{E}\left[E^2 du^2 + 2EF\,du\,dv + EG\,dv^2\right] \\ &= \frac{1}{E}\left[(E\,du + F\,dv)^2 + (EG - F^2)\,dv^2\right] \\ &= \frac{1}{E}\left[(E\,du + F\,dv)^2 + H^2\,dv^2\right] \\ &\geq 0 \quad \text{for all real values of } du \text{ and } dv \qquad \left[\because H^2 > 0\right] \end{aligned}$$

Again,

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0$$

gives

$$
\begin{aligned}
& (E\,du + F\,dv)^2 + H^2\,dv^2 = 0 \\
\text{i.e.,}\quad & (E\,du + F\,dv)^2 = 0 \quad \text{and} \quad H^2\,dv^2 = 0 \\
\text{i.e.,}\quad & E\,du + F\,dv = 0 \quad \text{and} \quad dv = 0 \quad [\because H^2 > 0] \\
\text{i.e.,}\quad & E\,du = 0 \quad \text{and} \quad dv = 0 \\
\text{i.e.,}\quad & du = 0 \quad \text{and} \quad dv = 0 \quad [\because E > 0]
\end{aligned}
$$

But both du and dv cannot vanish together. Hence metric,

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

is a positive definite quadratic form in du, dv.

Property 2. Invariance property. The metric is invariant under a transformation of parameters.

Proof. Let in the equation of the surface $\vec{r} = \vec{r}(u, v)$, the parameters u, v are transformed to the parameters u', v' by the relation (say)

$$u = \phi(u', v'), \quad v = \psi(u', v')$$

Thus,

$$\vec{r}_1' = \frac{\partial \vec{r}}{\partial u'} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial u'} = \vec{r}_1\frac{\partial u}{\partial u'} + \vec{r}_2\frac{\partial v}{\partial u'} \tag{1}$$

Similarly,

$$\vec{r}_2' = \frac{\partial \vec{r}}{\partial v'} = \vec{r}_1\frac{\partial u}{\partial v'} + \vec{r}_2\frac{\partial v}{\partial v'} \tag{2}$$

Again,

$$du = \frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv' \quad \text{and} \quad dv = \frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv' \tag{3}$$

Now,

$$\begin{aligned}
E' du'^2 + 2F' du' dv' + G' dv'^2 &= r_1'^2 du'^2 + 2(\vec{r}_1' \cdot \vec{r}_2') du' dv' + r_2'^2 dv'^2 \\
&= (\vec{r}_1' du' + \vec{r}_2' dv')^2 \\
&= \left[\left(\vec{r}_1 \frac{\partial u}{\partial u'} + \vec{r}_2 \frac{\partial v}{\partial u'} \right) du' + \left(\vec{r}_1 \frac{\partial u}{\partial v'} + \vec{r}_2 \frac{\partial v}{\partial v'} \right) dv' \right]^2 \\
&= \left[\vec{r}_1 \left(\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv' \right) + \vec{r}_2 \left(\frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv' \right) \right]^2 \\
&= (\vec{r}_1 du + \vec{r}_2 dv)^2 \\
&= r_1^2 du^2 + 2(\vec{r}_1 \cdot \vec{r}_2) du\, dv + r_2^2 dv^2 \\
&= E\, du^2 + 2F\, du\, dv + G\, dv^2
\end{aligned}$$

Hence, the metric is invariant.

Ex.1. Calculate first fundamental magnitudes for the surface

$$\vec{r} = [u\cos v,\ u\sin v,\ f(u)]$$

Solution. The given surface is

$$\vec{r} = [u\cos v,\ u\sin v,\ f(u)].$$

$$\vec{r}_1 = [\cos v,\ \sin v,\ f'(u)], \quad \vec{r}_2 = [-u\sin v,\ u\cos v,\ 0]$$

$$E = \vec{r}_1^{\,2} = \cos^2 v + \sin^2 v + f'^2 = 1 + f'^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -u\cos v\sin v + u\sin v\cos v + 0 = 0$$

$$G = \vec{r}_2^{\,2} = u^2\sin^2 v + u^2\cos^2 v + 0 = u^2$$

Ex.2. Calculate E, F, G, H for the paraboloid

$$x = u, \quad y = v, \quad z = u^2 - v^2.$$

Solution. The given surface is

$$\vec{r} = (u, v, u^2 - v^2).$$

We have

$$\vec{r}_1 = (1, 0, 2u), \quad \vec{r}_2 = (0, 1, -2v)$$

$$E = \vec{r}_1^{\,2} = 1 + 4u^2, \quad F = \vec{r}_1 \cdot \vec{r}_2 = 0 + 0 - 4uv = -4uv$$

$$G = \vec{r}_2^{\,2} = 1 + 4v^2$$

Also,

$$H = \sqrt{EG - F^2} = \sqrt{(1 + 4u^2)(1 + 4v^2) - 16u^2v^2}$$

$$= \sqrt{1 + 4u^2 + 4v^2}$$

Ex.3. Show that for the surface of revolution

$$x = u\cos v, \quad y = u\sin v, \quad z = f(u),$$

the parametric curves form an orthogonal system and

$$ds^2 = (1 + f'^2)\, du^2 + u^2 dv^2,$$

where dash denotes differentiation with respect to u.

Solution. The given surface is

$$\vec{r} = [u\cos v,\ u\sin v,\ f(u)].$$

$$\vec{r}_1 = [\cos v,\ \sin v,\ f'(u)], \quad \vec{r}_2 = [-u\sin v,\ u\cos v,\ 0]$$

$$E = \vec{r}_1^{\,2} = \cos^2 v + \sin^2 v + f'^2 = 1 + f'^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -u\cos v\sin v + u\sin v\cos v + 0 = 0$$

$$G = \vec{r}_2^{\,2} = u^2 \sin^2 v + u^2 \cos^2 v + 0 = u^2$$

Since $F = 0$, therefore the parametric curves are orthogonal. Again,

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = (1 + f'^2)\,du^2 + u^2\,dv^2.$$

Ex.4. Calculate E, F, G, H and the area corresponding to the domain $0 \le u \le 2\pi,\ 0 \le v \le 2\pi$ for the anchor ring

$$x = (b + a\cos u)\cos v, \quad y = (b + a\cos u)\sin v, \quad z = a\sin u.$$

The given surface is

$$\vec{r} = \big[(b + a\cos u)\cos v,\ (b + a\cos u)\sin v,\ a\sin u\big].$$

We have

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = (-a\sin u\cos v,\ -a\sin u\sin v,\ a\cos u),$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = (-(b + a\cos u)\sin v,\ (b + a\cos u)\cos v,\ 0).$$

Therefore,

$$E = \vec{r}_1^{\,2} = a^2 \sin^2 u \cos^2 v + a^2 \sin^2 u \sin^2 v + a^2 \cos^2 u = a^2,$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = a(b + a\cos u)\sin u \cos v \sin v - a(b + a\cos u)\sin u \sin v \cos v + 0 = 0,$$

$$G = \vec{r}_2^{\,2} = (b + a\cos u)^2 \sin^2 v + (b + a\cos u)^2 \cos^2 v = (b + a\cos u)^2.$$

Now,

$$H = \sqrt{EG - F^2} = \sqrt{a^2(b + a\cos u)^2} = a(b + a\cos u).$$

Elementary area at the point (u, v) is

$$H\,du\,dv.$$

$$\therefore \text{Required area} = \int_0^{2\pi}\int_0^{2\pi} H\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u)\,du\,dv$$

$$= \int_0^{2\pi} 2\pi a(b + a\cos u)\,du = 2\pi a \int_0^{2\pi} (b + a\cos u)\,du$$

$$= 2\pi a \left[bu + a\sin u\right]_0^{2\pi} = 2\pi a(2\pi b) = 4\pi^2 ab.$$

THANK YOU

Vist the website for notes

https://mathematicalexplorations.co.in

Subscribe to my YouTube Channel

Mathematical Explorations