UNIT 2 - Fluid Dynamics II

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MATHEMATICAL EXPLORATIONS

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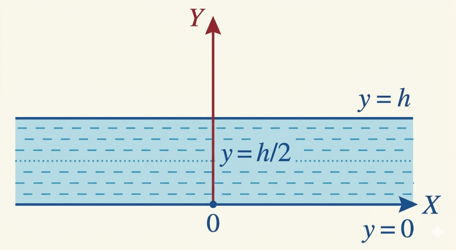
Steady Laminar Flow Between Two Parallel Plates (Plane Couette Flow)

Let us consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word 'infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two–dimensional (i.e. $\partial/\partial z = 0$).



Let the plates be long enough in the x–direction for the flow to be parallel. Since the plates are long in the x-direction, the flow becomes parallel to the plates. Therefore, no velocity across the plates and no velocity in the width direction i.e. the velocity components v and w to be zero everywhere. Moreover, the flow being steady, the flow variables are independent of time

$$\frac{\partial}{\partial t} = 0.$$

Furthermore, the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Substituting $v = 0$, $w = 0$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow \text{u does not depend on } x$$

so that

$$u = u(y).$$

Thus for the present problem we have

$$u = u(y), \qquad v = 0, \qquad w = 0, \qquad \frac{\partial}{\partial z} = 0, \qquad \frac{\partial}{\partial t} = 0. \tag{1}$$

For a Newtonian incompressible fluid without body forces, the Navier–Stokes equations are given by:

x–**direction**

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y–**direction**

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

For the present two–dimensional flow in absence of body forces, the Navier–Stokes equations for x and y directions are

$$0 = -\frac{\partial p}{\partial x} + \mu \left(\frac{d^2 u}{dy^2} \right) \tag{2}$$

$$0 = -\frac{\partial p}{\partial y} \tag{3}$$

Equation (3) shows that the pressure does not depend on y. Hence p is a function of x alone and therefore equation (2) reduces to

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu} \left(\frac{dp}{dx} \right). \tag{4}$$

Differentiating both sides of equation (4) with respect to x,

$$0 = \frac{1}{\mu}\frac{d^2p}{dx^2}$$

or

$$\frac{d}{dx}\left(\frac{dp}{dx}\right) = 0.$$

so that

$$\frac{dp}{dx} = \text{const.} = P \text{ (say)}. \tag{5}$$

Then, equation (4) reduces to

$$\frac{d^2u}{dy^2} = \frac{P}{\mu}. \tag{6}$$

Integrating (6),

$$\frac{du}{dy} = \frac{Py}{\mu} + A. \tag{7}$$

Integrating (7),

$$u = Ay + B + \frac{P}{2\mu}y^2. \tag{8}$$

where A and B are arbitrary constants to be determined by the boundary conditions of the problem under consideration.

For the plane Couette flow there is no pressure gradient, $P = 0$. Again, the plate $y = 0$ is kept at rest and the plate $y = h$ is allowed to move with velocity U. Then the no–slip condition gives rise to the boundary conditions

$$u = 0 \quad \text{at} \quad y = 0, \qquad u = U \quad \text{at} \quad y = h. \tag{9}$$

Using (9) in (8) yields

$$0 = B \quad \text{and} \quad U = Ah + B$$

so that

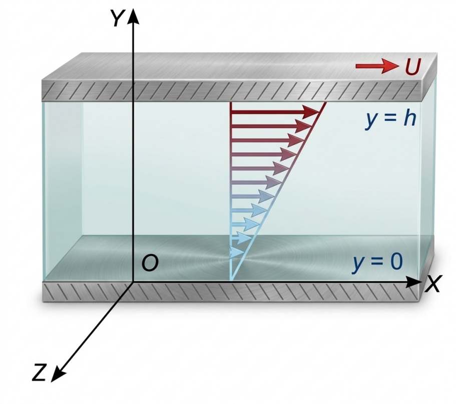
$$B = 0 \qquad \text{and} \qquad A = \frac{U}{h}. \tag{10}$$

Using (10) in (8), we obtain

$$u = \frac{U}{h} y$$

which is the velocity profile of Coutte flow. The velocity distribution is linear which means velocity increases uniformly from bottom plate to top plate as shown in the adjoining figure. Now the skin friction (or drag per unit area, i.e., the shearing stress at the plates) σ_{yx} is given by

$$\sigma_{yx} = \mu \left(\frac{du}{dy} \right) = \frac{\mu U}{h}$$

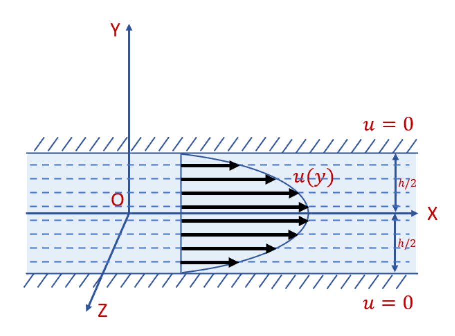


Plane Poiseuille Flow

Let us consider the steady laminar flow of a viscous incompressible fluid between two infinite parallel plates separated by a distance h.

Let the coordinate system be chosen as follows:

- The x-axis is taken along the direction of the flow and lies midway between the two plates.
- The y-axis is perpendicular to the plates.
- The z-axis is along the width of the plates.



The term *infinite plates* means that the width of the plates in the z-direction is very large compared to the separation h. Therefore, the flow can be treated as two–dimensional and

$$\frac{\partial}{\partial z} = 0.$$

The plates are assumed to be sufficiently long in the x-direction so that the flow becomes fully developed and parallel.
For this flow, the velocity components in the y and z directions are zero everywhere. Hence,

$$v = 0, \qquad w = 0.$$

Since the flow is steady, all flow variables are independent of time, therefore

$$\frac{\partial}{\partial t} = 0.$$

The continuity equation for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Using $v = 0$ and $w = 0$, the above equation reduces to

$$\frac{\partial u}{\partial x} = 0.$$

Thus the velocity component u depends only on the transverse coordinate y, i.e.,

$$u = u(y).$$

Hence, for the present problem,

$$u = u(y), \qquad v = 0, \qquad w = 0, \qquad \frac{\partial}{\partial z} = 0, \qquad \frac{\partial}{\partial t} = 0. \tag{1}$$

For an incompressible fluid, the Navier–Stokes equations in Cartesian coordinates are given as follows.

x–**direction:**

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho F_x$$

y–**direction:**

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho F_y$$

For the present two–dimensional flow in the absence of body forces, the Navier–Stokes equations in the x and y directions reduce to

$$0=-\frac{\partial p}{\partial x}+\mu\frac{d^2 u}{dy^2} \tag{2}$$

$$0 = -\frac{\partial p}{\partial y} \quad (3)$$

Equation (3) shows that the pressure does not depend on y. Therefore the pressure p is a function of x only. Hence equation (2) becomes

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \quad (4)$$

Differentiating equation (4) with respect to x, we obtain

$$0 = \frac{1}{\mu}\frac{d^2p}{dx^2}$$

or equivalently

$$\frac{d}{dx}\left(\frac{dp}{dx}\right) = 0$$

which implies that

$$\frac{dp}{dx} = \text{constant} = P \tag{5}$$

Substituting this result into equation (4), we get

$$\frac{d^2u}{dy^2} = \frac{P}{\mu} \tag{6}$$

Integrating equation (6) with respect to y, we obtain

$$\frac{du}{dy} = \frac{P}{\mu}y + A \tag{7}$$

where A is a constant of integration.
Integrating once again with respect to y, we obtain

$$u = Ay + B + \frac{P}{2\mu}y^2 \tag{8}$$

where B is another constant of integration.
Let the plates be located at

$$y = -\frac{h}{2}, \qquad y = \frac{h}{2},$$

so that the x–axis lies along the centreline between the plates.
Using the no–slip condition at the walls, the velocity must vanish at both plates.
Therefore, the boundary conditions are

$$u = 0 \quad \text{at} \quad y = -\frac{h}{2},$$

$$u = 0 \quad \text{at} \quad y = \frac{h}{2}. \tag{9}$$

From the general velocity solution obtained earlier (Eq. (8)),

$$u = Ay + B + \frac{P}{2\mu}y^2,$$

applying the boundary conditions gives

$$0 = -\frac{Ah}{2} + B + \frac{Ph^2}{8\mu},$$

$$0 = \frac{Ah}{2} + B + \frac{Ph^2}{8\mu}.$$

Solving these two equations yields

$$A = 0,$$

$$B = -\frac{h^2 P}{8\mu}.$$

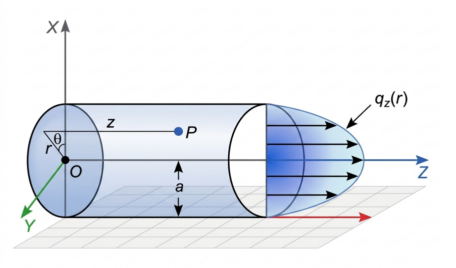
Substituting these constants into the velocity expression, Eq. (8) reduces to

$$u = -\frac{h^2 P}{8\mu} \left[1 - 4 \left(\frac{y}{h} \right)^2 \right]. \tag{10}$$

This expression shows that the velocity distribution across the channel is **parabolic**, with the maximum velocity occurring at the centreline $y = 0$.

Flow Through a Circular Pipe: The Hagen–Poiseuille Flow

Let us consider the laminar steady flow of an incompressible fluid through an infinitely long circular pipe of radius a. The flow is assumed to occur without body forces and possesses axial symmetry as shown in the figure.



For the present problem we employ cylindrical coordinates (r, θ, z). Let z be the direction of flow along the axis of the pipe.

Because of axial symmetry of the flow,

$$q_r = 0, \qquad q_\theta = 0$$

that is, the radial and tangential velocity components vanish.

Further, the axial velocity component is independent of the angular coordinate θ, so that

$$q_z = q_z(r)$$

The continuity equation for steady incompressible flow in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

Since

$$q_r = 0, \qquad q_\theta = 0,$$

the above equation reduces to

$$\frac{\partial q_z}{\partial z} = 0$$

This shows that the axial velocity q_z is also independent of z. Hence the velocity is a function of the radial coordinate alone,

$$q_z = q_z(r)$$

Thus, for the present problem,

$$q_r = 0, \qquad q_\theta = 0, \qquad q_z = q_z(r) \tag{1}$$

The momentum equations in cylindrical coordinates are given by
Radial component

$$\rho \left(\frac{\partial q_r}{\partial t} + q_r \frac{\partial q_r}{\partial r} + \frac{q_\theta}{r} \frac{\partial q_r}{\partial \theta} - \frac{q_\theta^2}{r} + q_z \frac{\partial q_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left[\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2} \frac{\partial q_\theta}{\partial \theta} \right]$$

Tangential component

$$\rho\left(\frac{\partial q_\theta}{\partial t} + q_r\frac{\partial q_\theta}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r q_\theta}{r} + q_z\frac{\partial q_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 q_\theta - \frac{q_\theta}{r^2} + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta}\right]$$

Axial component

$$\rho\left(\frac{\partial q_z}{\partial t} + q_r\frac{\partial q_z}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_z}{\partial \theta} + q_z\frac{\partial q_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 q_z$$

where

$$\nabla^2 q_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 q_z}{\partial \theta^2} + \frac{\partial^2 q_z}{\partial z^2}$$

For steady axisymmetric flow of an incompressible fluid with velocity components given by Eq. (1), the Navier–Stokes equations in cylindrical coordinates reduce to

$$0 = -\frac{\partial p}{\partial r}, \tag{2}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{3}$$

$$0 = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r} \right) \right]. \tag{4}$$

Equations (2) and (3) show that the pressure p is independent of r and θ. Hence the pressure is a function of z alone, i.e.,

$$p = p(z).$$

Further, from Eq. (1), the axial velocity component is a function of r alone,

$$q_z = q_z(r).$$

Therefore Eq. (4) may be rewritten as

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right)\right]=\frac{dp}{dz}. \tag{5}$$

Differentiating both sides of Eq. (5) with respect to z, we obtain

$$0=\frac{d^2p}{dz^2}$$

or

$$\frac{d}{dz}\left(\frac{dp}{dz}\right)=0. \tag{6}$$

Hence,

$$\frac{dp}{dz}=\text{constant}=-P \quad \text{(say)}. \tag{7}$$

We may take

$$P = \frac{p_2 - p_1}{l},$$

where p_1 and p_2 denote the values of pressure at the ends of a length l of the circular pipe.
Let us now write

$$q_z = u.$$

Substituting Eq. (7) into Eq. (5), we obtain

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{Pr}{\mu}. \tag{8}$$

Integrating Eq. (8) with respect to r, we obtain

$$r\frac{du}{dr} = \frac{Pr^2}{2\mu} + A, \tag{9}$$

where A is a constant of integration.
Dividing by r,

$$\frac{du}{dr} = \frac{Pr}{2\mu} + \frac{A}{r}.$$

Integrating again,

$$u = \frac{Pr^2}{4\mu} + A \log r + B, \tag{10}$$

where A and B are constants of integration.
The velocity must remain finite on the axis of the pipe ($r = 0$). Since $\log r \to -\infty$ as $r \to 0$, the term $A \log r$ would make the velocity infinite unless

$$A = 0.$$

Thus Eq. (10) reduces to

$$u = \frac{Pr^2}{4\mu} + B. \tag{11}$$

Since the circular boundary of the pipe is at rest, the no-slip condition gives

$$u = 0 \quad \text{at} \quad r = a. \tag{12}$$

Substituting Eq. (12) into Eq. (11), we obtain

$$B = -\frac{Pa^2}{4\mu}.$$

Hence the velocity distribution becomes

$$u = -\frac{Pa^2}{4\mu}\left[1 - \left(\frac{r}{a}\right)^2\right]. \tag{13}$$

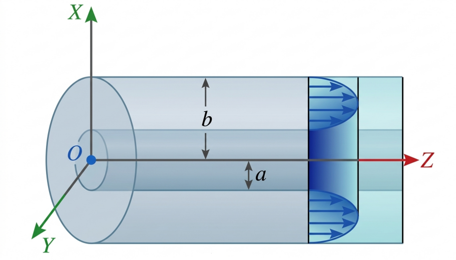
This expression represents a **parabolic velocity distribution** across the pipe section, which is characteristic of Hagen–Poiseuille flow.

Laminar Steady Flow Between Two Coaxial Circular Cylinders

For the present problem, we consider the governing equations in cylindrical coordinates (r, θ, z). Let z be the direction of flow along the axis of the pipe.

Clearly, the radial and tangential velocity components are zero, that is,

$$q_r = 0, \qquad q_\theta = 0.$$



Due to axial symmetry of the flow, the axial velocity component q_z is independent of the angular coordinate θ.

The equation of continuity for steady incompressible flow in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0.$$

Since

$$q_r = 0, \qquad q_\theta = 0,$$

the continuity equation reduces to

$$\frac{\partial q_z}{\partial z} = 0,$$

showing that q_z is independent of z also. Hence the axial velocity depends only on the radial coordinate, i.e.,

$$q_z = q_z(r). \tag{1}$$

Thus for the present problem,

$$q_r = 0, \qquad q_\theta = 0, \qquad q_z = q_z(r).$$

For an incompressible Newtonian fluid, the Navier–Stokes equations are:

Radial component

$$\rho\left(\frac{\partial q_r}{\partial t} + q_r\frac{\partial q_r}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta^2}{r} + q_z\frac{\partial q_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}\right)$$

Angular component

$$\rho\left(\frac{\partial q_\theta}{\partial t} + q_r\frac{\partial q_\theta}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r q_\theta}{r} + q_z\frac{\partial q_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 q_\theta - \frac{q_\theta}{r^2} + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta}\right)$$

Axial (z) component

$$\rho\left(\frac{\partial q_z}{\partial t} + q_r\frac{\partial q_z}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_z}{\partial \theta} + q_z\frac{\partial q_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 q_z$$

For steady axisymmetric flow of an incompressible fluid with velocity components given by Eq. (1), the Navier–Stokes equations in cylindrical coordinates reduce to

$$0 = -\frac{\partial p}{\partial r}, \tag{2}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{3}$$

$$0 = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)\right]. \tag{4}$$

These equations govern the laminar steady flow of an incompressible fluid between two coaxial circular cylinders.

(2) and (3) show that p is independent of r and θ. Thus p is a function of z alone. Further q_z is a function of r alone by (1). Hence (4) may be re-written as

$$\mu \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dq_z}{dr} \right) \right] = \frac{dp}{dz} \tag{5}$$

Differentiating both sides of (5) w.r.t. z, we find

$$0 = \frac{d^2 p}{dz^2} \quad \text{or} \quad \frac{d}{dz} \left(\frac{dp}{dz} \right) = 0$$

so that

$$\frac{dp}{dz} = \text{const.} = P \text{ (say)}. \tag{6}$$

We may take

$$P = \frac{p_2 - p_1}{l}, \tag{7}$$

where p_1, p_2 denote the values of p, at the ends of a length l of the circular pipe. In what follows, we now write $q_z = u$. Then, using (6), (5) reduces to

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{Pr}{\mu} \tag{8}$$

Integrating (8),

$$r\frac{du}{dr} = \frac{Pr^2}{2\mu} + A \quad \text{or} \quad \frac{du}{dr} = \frac{Pr}{2\mu} + \frac{A}{r} \tag{9}$$

Integrating (9),

$$u = \frac{Pr^2}{4\mu} + A\log r + B, \tag{10}$$

where the constants A and B are to be found by using the boundary conditions. Now u must be finite on the axis of the tube (where $r = 0$). So we must take $A = 0$

in (10) because otherwise u would become infinite when $r = 0$. Thus (10) reduces to

$$u = \frac{Pr^2}{4\mu} + B. \tag{11}$$

Since the circular boundary of the tube is at rest, the no-slip condition at the wall gives rise to the following boundary condition

$$u = 0 \quad \text{at} \quad r = a. \tag{12}$$

Using (12), (11) gives $B = -\frac{Pa^2}{4\mu}$. Hence (11) becomes

$$u = -\frac{Pa^2}{4\mu}\left\{1 - \left(\frac{r}{a}\right)^2\right\}, \tag{13}$$

which has the form of a paraboloid of revolution.

THANK YOU

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