UNIT 2 - Fluid Dynamics II

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MATHEMATICAL EXPLORATIONS

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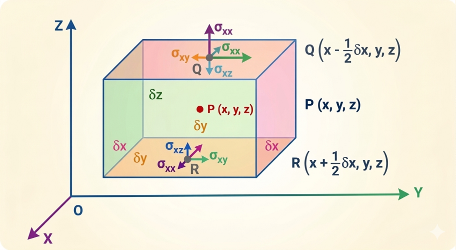
Navier-Stoke's Equation of Motion of a Viscous Fluid:

Let us consider the motion of a small rectangular parallelepiped of viscous fluid whose centre is at $P(x, y, z)$. Let the lengths of its edges parallel to the coordinate axes be δx, δy, and δz respectively.



The volume of the fluid element is

$$\delta V = \delta x \, \delta y \, \delta z.$$

If ρ is the density of the fluid, then the mass of the element is

$$m = \rho \, \delta x \, \delta y \, \delta z.$$

The fluid element moves together with the surrounding fluid, and therefore its mass remains constant.

In a viscous fluid, internal forces act across the surfaces of the fluid element. These forces are described in terms of the stress components σ_{ij}, where

- the first suffix i denotes the direction of the normal to the surface,
- the second suffix j denotes the direction of the force component.

Thus the stresses acting on a surface perpendicular to the x-axis are

$$\sigma_{xx}, \qquad \sigma_{xy}, \qquad \sigma_{xz}.$$

Let us consider the face of the element perpendicular to the x-axis having area

$$dA = dy\,dz$$

The outward unit normal to this surface is $\hat{i}$. Hence the components of the forces acting on this face are obtained by multiplying the stresses by the area.
Therefore the force components are

$$[\sigma_{xx}\, dy\, dz,\ \sigma_{xy}\, dy\, dz,\ \sigma_{xz}\, dy\, dz].$$

Let the parallel face located at

$$R\left(x + \frac{1}{2}\delta x,\ y,\ z\right).$$

Since the stress components vary from point to point in the fluid, their values at this point can be obtained by expanding them about the point $P(x, y, z)$ using the Taylor series expansion and neglecting higher order terms.
Thus we obtain

$$\sigma_{xx}\left(x + \frac{1}{2}\delta x, y, z\right) = \sigma_{xx} + \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xx}}{\partial x}\right),$$

$$\sigma_{xy}\left(x + \frac{1}{2}\delta x, y, z\right) = \sigma_{xy} + \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xy}}{\partial x}\right),$$

$$\sigma_{xz}\left(x+\frac{1}{2}\delta x, y, z\right) = \sigma_{xz} + \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xz}}{\partial x}\right).$$

Multiplying these stresses by the area $dy\,dz$, the force components acting on this face are

$$\left[\left\{\sigma_{xx} + \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xx}}{\partial x}\right)\right\} dy\,dz, \left\{\sigma_{xy} + \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xy}}{\partial x}\right)\right\} dy\,dz, \left\{\sigma_{xz} + \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xz}}{\partial x}\right)\right\} dy\,dz\right]$$

Now consider the opposite face passing through

$$Q\left(x-\frac{1}{2}\delta x,\ y,\ z\right).$$

The outward unit normal to this surface is $-\hat{i}$. Therefore the stresses act in the opposite direction.

Using Taylor expansion about the point $P(x, y, z)$ we obtain

$$\sigma_{xx}\left(x - \frac{1}{2}\delta x, y, z\right) = \sigma_{xx} - \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xx}}{\partial x}\right),$$

$$\sigma_{xy}\left(x - \frac{1}{2}\delta x, y, z\right) = \sigma_{xy} - \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xy}}{\partial x}\right),$$

$$\sigma_{xz}\left(x - \frac{1}{2}\delta x, y, z\right) = \sigma_{xz} - \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xz}}{\partial x}\right).$$

Multiplying by the area $dy\,dz$ and introducing the negative sign due to the outward normal $-\hat{i}$, the force components on this face are

$$\left[-\left\{\sigma_{xx} - \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xx}}{\partial x}\right)\right\} dy\,dz, -\left\{\sigma_{xy} - \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xy}}{\partial x}\right)\right\} dy\,dz, -\left\{\sigma_{xz} - \frac{1}{2}\delta x\left(\frac{\partial \sigma_{xz}}{\partial x}\right)\right\} d\right.$$

Let the forces acting on the two faces of the cuboid perpendicular to the x–axis passing through the points

$$R\left(x+\frac{1}{2}\delta x, y, z\right), \qquad Q\left(x-\frac{1}{2}\delta x, y, z\right).$$

Each face has force components due to the stresses

$$\sigma_{xx}, \quad \sigma_{xy}, \quad \sigma_{xz}$$

acting on an area

$$dy\, dz$$

Force on the right face:

$$\left(\sigma_{xx}+\frac{1}{2}\delta x\frac{\partial \sigma_{xx}}{\partial x}\right) dy\, dz$$

Force on the left face:

$$-\left(\sigma_{xx} - \frac{1}{2}\delta x \frac{\partial \sigma_{xx}}{\partial x}\right) dy\, dz$$

Adding the two forces,

$$= \delta x \frac{\partial \sigma_{xx}}{\partial x} dy\, dz$$

Thus the resultant force component in the x-direction becomes

$$\frac{\partial \sigma_{xx}}{\partial x}\, \delta x\, dy\, dz$$

Similarly,

$$\frac{\partial \sigma_{xy}}{\partial x}\, \delta x\, dy\, dz.$$

$$\frac{\partial \sigma_{xz}}{\partial x}\,\delta x\,dy\,dz.$$

Thus the two faces perpendicular to the x-axis produce a single force at P having components

$$\left[\frac{\partial \sigma_{xx}}{\partial x}, \frac{\partial \sigma_{xy}}{\partial x}, \frac{\partial \sigma_{xz}}{\partial x}\right]\delta x\,dy\,dz. \tag{1}$$

Although the forces combine into a resultant force, their lines of action are not exactly through P. Therefore they also produce moments (couples).
Moment about the OY-axis:

$$-\,\sigma_{xz}\,\delta x\,dy\,dz$$

Moment about the OZ-axis:

$$+\sigma_{xy}\,\delta x\,dy\,dz \tag{2}$$

Now consider the two faces located at

$$y+\frac{1}{2}\delta y, \qquad y-\frac{1}{2}\delta y.$$

The stresses acting on these surfaces are

$$\sigma_{yx}, \quad \sigma_{yy}, \quad \sigma_{yz}$$

Using the same Taylor expansion procedure, the resultant force becomes

$$\left[\frac{\partial\sigma_{yx}}{\partial y}, \frac{\partial\sigma_{yy}}{\partial y}, \frac{\partial\sigma_{yz}}{\partial y}\right]\delta x\,\delta y\,\delta z. \tag{3}$$

Moment about OZ:

$$-\sigma_{yx}\,\delta x\,\delta y\,\delta z$$

Moment about OX:

$$+\sigma_{yz}\,\delta x\,\delta y\,\delta z \tag{4}$$

Now consider the two faces

$$z+\frac{1}{2}\delta z, \qquad z-\frac{1}{2}\delta z.$$

The stresses acting here are

$$\sigma_{zx}, \quad \sigma_{zy}, \quad \sigma_{zz}.$$

Using Taylor expansion, the resultant force becomes

$$\left[\frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{zz}}{\partial z}\right] \delta x\, \delta y\, \delta z. \tag{5}$$

Moment about OX:

$$-\sigma_{zy}\, \delta x\, \delta y\, \delta z$$

Moment about OY:

$$+\sigma_{zx}\, \delta x\, \delta y\, \delta z \tag{6}$$

Adding the contributions from the three pairs of faces, the total surface force at P becomes

$$\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z},\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z},\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right] \delta x\, \delta y\, \delta z. \tag{7}$$

These terms represent the divergence of the stress tensor.
Adding all couples produced by the six faces gives

$$[(\sigma_{yz} - \sigma_{zy}), (\sigma_{zx} - \sigma_{xz}), (\sigma_{xy} - \sigma_{yx})]\, \delta x\, \delta y\, \delta z. \tag{8}$$

For ordinary fluids, the stress tensor is symmetric, that is

$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{zy}, \qquad \sigma_{zx} = \sigma_{xz}.$$

Hence the resultant couple vanishes.
Apart from surface stresses, the fluid also experiences external body forces such as gravity or electromagnetic forces.
Let the body force per unit mass be

$$(X, Y, Z).$$

Then the force on the fluid element of mass

$$\rho\,\delta x\,\delta y\,\delta z$$

is

$$[X, Y, Z]\,\rho\,\delta x\,\delta y\,\delta z. \tag{9}$$

Adding the surface force and the body force, the total force component in the x-direction becomes

$$\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right]\delta x\,\delta y\,\delta z + \rho X\,\delta x\,\delta y\,\delta z. \tag{10}$$

Let us consider $\vec{q}(u, v, w)$ be the velocity at the point P at any time t, then the equation of motion along the i-direction is obtained

$$(\rho\,\delta x\,\delta y\,\delta z)\frac{du}{dt} = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right]\delta x\,\delta y\,\delta z + \rho X\,\delta x\,\delta y\,\delta z. \tag{11}$$

or

$$\frac{du}{dt} = X + \frac{1}{\rho}\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right].$$

Similarly the equations of motion in the direction of $\hat{j}$ and $\hat{k}$ are given by

$$\frac{dv}{dt} = Y + \frac{1}{\rho}\left[\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right],$$

and

$$\frac{dw}{dt} = Z + \frac{1}{\rho}\left[\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right]. \tag{12}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

The stress tensor for a viscous fluid is given by

$$\sigma_{ij} = (-p + \lambda\Delta)\delta_{ij} + \mu e_{ij}$$

Thus,

$$\sigma_{xx} = (-p + \lambda\Delta)\delta_{xx} + \mu e_{xx} = -p + \lambda\Delta + 2\mu\left(\frac{\partial u}{\partial x}\right),$$

and

$$\sigma_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad \sigma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \sigma_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right). \tag{13}$$

From (12) and (13), we have

$$\frac{du}{dt} = X + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(-p + \lambda\Delta + 2\mu\frac{\partial u}{\partial x}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right].$$

or

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\lambda}{\rho}\frac{\partial \Delta}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right] + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right].$$

or

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\nu + \frac{\lambda}{\rho}\right)\frac{\partial \Delta}{\partial x} + \nu \nabla^2 u.$$

Some fluids have a volume viscosity that measures their resistance to volume changes. In the case of a fluid with a volume viscosity, the expression $\left[\lambda + \left(\frac{2}{3}\right)\mu\right]$ is not zero. In the case of a compressible flow

$$\lambda = -\left(\frac{2}{3}\right)\mu$$

or

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\nu\frac{\partial \Delta}{\partial x} + \nu\nabla^2 u$$

Thus the equations of motion along the coordinate-axes are given by

$$\begin{aligned}\frac{du}{dt} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\nu\frac{\partial \Delta}{\partial x} + \nu\nabla^2 u,\\ \frac{dv}{dt} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{3}\nu\frac{\partial \Delta}{\partial y} + \nu\nabla^2 v,\\ \frac{dw}{dt} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{3}\nu\frac{\partial \Delta}{\partial z} + \nu\nabla^2 w \end{aligned} \tag{14}$$

These equations can be written in the form

$$
\begin{aligned}
\rho \frac{Du}{Dt} &= \rho X - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)\right] \\
&\quad + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \\
\rho \frac{Dv}{Dt} &= \rho Y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)\right] \\
&\quad + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
\rho \frac{Dw}{Dt} &= \rho Z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)\right] \\
&\quad + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]
\end{aligned}
\tag{15}
$$

which are the general Navier–Stokes equations of motion in Cartesian coordinates.

THANK YOU

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