Waves - Fluid Dynamics II

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MATHEMATICAL EXPLORATIONS

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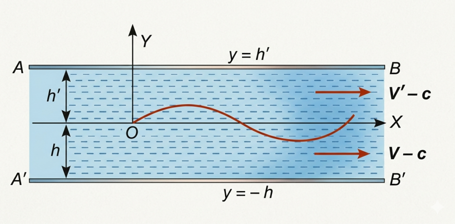
Waves at the interface (i.e. common surface) of two liquids:

Let a liquid of density ρ' and depth h' move with velocity V' over another liquid of density ρ and depth h moving in the same direction with velocity V; the liquids being bounded above and below by two fixed horizontal planes AB and $A'B'$.

Let c be the velocity of propagation of oscillatory waves at the interface of two liquids in the direction in which the liquids are moving.



Let the x-axis be taken in the direction in which the undisturbed interface (i.e. common surface of two liquids) and y-axis vertically upwards. We make the motion steady by superposing on the whole mass the velocity $-c$. Thus the wave profile is reduced to rest in space and the new velocities of liquids become $V' - c$ and $V - c$ as shown in figure.

The velocity potential and stream function for the lower liquid moving with $-(V-c)$ in the negative direction of x-axis are given by

$$\phi = -(V - c)x + D\cosh m(y + h)\cos mx, \tag{1}$$
$$\psi = -(V - c)y - D\sinh m(y + h)\sin mx. \tag{2}$$

Similar expression for the upper liquid may be deduced from (1) and (2) by replacing V by V' and h by h'. Thus, we get

$$\phi' = -(V' - c)x + D'\cosh m(y - h')\cos mx, \tag{3}$$
$$\psi' = -(V' - c)y - D'\sinh m(y - h')\sin mx. \tag{4}$$

Clearly the above expression for ψ and ψ' make the boundaries $y = -h$, $y = h'$ streamlines.
Let

$$\eta = a\sin mx. \tag{5}$$

represent the displacement of the interface. If the liquids do not separate, then (5) must be a streamline for both surfaces. This condition is satisfied by assuming the streamline to be $\psi = \psi' = 0$. Neglecting the squares of small quantities (e.g. a^2), we thus obtain

$$-(V-c)a - D\sinh mh = 0, \tag{6}$$

$$-(V'-c)a + D'\sinh mh' = 0. \tag{7}$$

From Bernoulli's equations, we obtain

$$\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} = \text{const.} \tag{8}$$

$$\frac{p'}{\rho'} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi'}{\partial x}\right)^2 + \left(\frac{\partial\phi'}{\partial y}\right)^2\right\} = \text{const.} \tag{9}$$

But at the interface $y = \eta = a \sin mx$. Hence neglecting a^2, (8) and (9) give

$$\frac{p}{\rho} + ga \sin mx + \frac{1}{2}(V - c)^2 (1 - 2am \coth mh \sin mx) = \text{const.},$$
$$\frac{p'}{\rho'} + ga \sin mx + \frac{1}{2}(V' - c)^2 \left(1 + 2am \coth mh' \sin mx\right) = \text{const.}$$

Since the pressure is continuous across the interface, putting $p = p'$ in above equations, subtracting and then equating to zero the coefficient of $\sin mx$, we obtain

$$g(\rho - \rho') = (V - c)^2 m\rho \coth mh + (V' - c)^2 m\rho' \coth mh'. \tag{10}$$

Equation (10) determines the velocity of propagation c of waves of wave length $2\pi/m$ at the interface. We can also treat (10) as the condition for stationary waves at the interface of two streams whose velocities are $V - c$ and $V' - c$.

Cor. 1. When the liquids are at rest (i.e. $V = V' = 0$), the wave velocity is given by

$$c^2 = \frac{g}{m} \frac{\rho - \rho'}{\rho \coth mh + \rho' \coth mh'}. \tag{11}$$

Since there is no real value of c when $\rho' > \rho$, (11) shows that when $\rho' > \rho$ the equilibrium position is unstable.

Cor. 2. Let the liquids be at rest and the depths of both liquids be so large compared to the wave length that we may take $\coth mh = \coth mh' = 1$. Then we have

$$c^2 = \frac{g}{m} \frac{\rho - \rho'}{\rho + \rho'} \quad \text{or} \quad c^2 = \frac{g\lambda}{2\pi} \frac{\rho - \rho'}{\rho + \rho'}. \tag{12}$$

Cor. 3. Let the liquids be at rest and the upper fluid be air.

Let $s = \rho'/\rho$ be the specific gravity of air. Suppose the depth of air be infinite so that $\coth mh' \to 1$ as $h' \to \infty$. Then (11) reduces to

$$c^2 = \frac{g}{m}\frac{\rho - \rho'}{\rho \coth mh + \rho'} = \frac{g}{m}\frac{1-s}{\coth mh + s}$$

$$= \frac{g(1-s)}{m \coth mh}(1 + s\tanh mh)^{-1} = \frac{g \tanh mh}{m}(1-s)(1 - s\tanh mh)$$

(expanding by Binomial theorem and neglecting s^2 and higher powers of s). Simplifying, neglecting s^2 and writing $s = \rho'/\rho$ we get

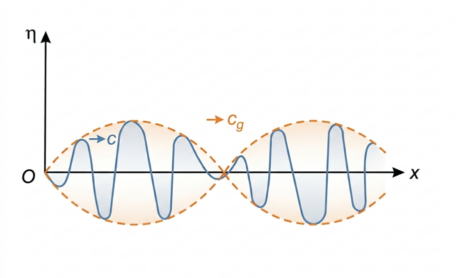
$$c^2 = \left(\frac{g}{m}\right)\tanh mh \left\{1 - \left(\frac{\rho'}{\rho}\right)(1 + \tanh mh)\right\} \quad \text{approximately.} \tag{13}$$

Cor. 4. Let the velocities V, V' make angles α, α' with the direction of c. But the components $v' \sin \alpha'$ and $v \sin \alpha$ (perpendicular to the direction of c) do not affect the value of c. Hence the required value of c can be obtained by replacing V and V' by $V\cos\alpha$ and $V'\cos\alpha'$ in (10). Thus, we get

$$g(\rho-\rho') = m\rho(V\cos\alpha - c)^2\coth mh + m\rho'(V'\cos\alpha' - c)^2\coth mh'. \tag{14}$$

Group velocity:

When a stone is dropped into a pond or a boat starts moving on the surface of still water, a local disturbance takes place. This gives rise to a wave which can be analysed into a set of simple harmonic components each of different wave length. Since the velocity of propagation depends upon the wave length, the waves of different wave lengths will be gradually sorted out into groups of waves of approximately the same wave length. Since the waves in front pass out of the group and new waves enter the group from behind, the energy within the group does not change.



Let us consider the superposition of two simple harmonic waves of the same amplitude and slightly different wave lengths.

$$\eta_1 = a\sin(mx - nt)$$

and

$$\eta_2 = a\sin\{(m+\delta m)x - (n+\delta n)t\} \tag{1}$$

The resulting disturbance is given by

$$\eta = \eta_1 + \eta_2$$

i.e.

$$\eta = 2a\cos\left\{\frac{1}{2}(x\,\delta m - t\,\delta n)\right\}\sin(mx - nt) = A\sin(mx - nt) \tag{2}$$

where

$$A = 2a\cos\left\{\frac{1}{2}(x\,\delta m - t\,\delta n)\right\} \tag{3}$$

Equation (2) shows that the resulting disturbance is a progressive sine wave whose amplitude A is not constant but is itself varying as a wave of velocity $c_g = \delta n / \delta m$. This velocity is known as the *group velocity*.

Since the velocity of propagation of a single wave is

$$c = \frac{n}{m},$$

we have

$$c_g = \frac{dn}{dm} = \frac{d(cm)}{dm} = c + m\frac{dc}{dm} \tag{4}$$

But

$$\lambda = \frac{2\pi}{m}$$

so that

$$\frac{d\lambda}{dm} = -\frac{2\pi}{m^2} \tag{5}$$

Using (5), (4) may be re-written as

$$c_g = c + m\frac{dc}{d\lambda}\frac{d\lambda}{dm} = c - \lambda\frac{dc}{d\lambda} \tag{6}$$

For the case of waves on the surface of liquid of depth h, we have

$$c^2 = \frac{g}{m}\tanh mh \tag{7}$$

From (4) and (7), we have

$$c_g = c\left(1 + \frac{m}{2c^2}\frac{dc^2}{dm}\right) = \frac{1}{2}c\left(1 + \frac{2mh}{\sin 2mh}\right) \tag{8}$$

so that the ratio of the group velocity to the wave velocity is given by

$$\frac{c_g}{c} = \frac{1}{2}\left(1 + \frac{2mh}{\sin 2mh}\right).$$

giving

$$c_g = \frac{1}{2}c\left(1 + 2mh\,\text{cosech}\,2mh\right) \tag{9}$$

When h is small (e.g. consider shallow water) compared with the wave length, $c_g/c = 1$ so that group velocity for shallow water is equal to the wave velocity. Again, when $h \to \infty$ (e.g. consider deep sea waves), $c_g/c = 1/2$ i.e. $c_g = c/2$. Thus, the group velocity for deep sea waves is half the wave velocity.

Rate of transmission of energy in simple harmonic surface waves. Dynamical significance of group velocity:

In a simple harmonic train of surface waves, energy crosses a fixed vertical plane perpendicular to the direction of propagation at an average rate equal to group velocity.

Proof. Consider a vertical section of the liquid (of depth h) at right angles to the direction of propagation. Then the rate of transmission of energy is calculated by determining the rate at which the pressure on one side of the chosen section is doing work on the liquid on the other side.

$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx-nt). \tag{1}$$

Again neglecting squares of small quantities the variable part of the pressure is given by

$$\delta p = \rho\left(\frac{\partial \phi}{\partial t}\right). \tag{2}$$

and the horizontal velocity u is given by

$$u = -\frac{\partial \phi}{\partial x}. \tag{3}$$

Hence the rate at which work is being done on the fluid to the right of x is given by

$$W = \int_{-h}^{0} \delta p \, u \, dy = -\int_{-h}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} dy = \frac{g^2 \rho a^2}{n} \frac{\sin^2(mx - nt)}{\cosh^2 mh} \int_{-h}^{0} \cosh^2 m(y+h) \, dy,$$

by (1), (2) and (3).
Then

$$W = \frac{g^2 \rho a^2 m \sin^2(mx - nt)}{n \cosh^2 mh} \left(\frac{\sinh 2mh}{4m} + \frac{h}{2} \right). \tag{5}$$

Now we have,

$$n^2 = gm \tanh mh.$$

Hence (5) reduces to

$$W = \frac{1}{2} g\rho a^2 \frac{n}{m}(1 + 2mh \operatorname{cosech} 2mh) \sin^2(mx - nt). \tag{6}$$

The average value of $\sin^2(mx - nt)$ over a period is $1/2$. Hence the average rate of work done is given from (6) by

$$W = \frac{1}{4} g\rho a^2 \frac{n}{m}(1 + 2mh \operatorname{cosech} 2mh). \tag{7}$$

But group velocity c_g is given by

$$c_g = \frac{c}{2}(1 + 2mh \operatorname{cosech} 2mh). \tag{8}$$

Since $n/m = c$, (7) and (8) give

$$W = \frac{g\rho a^2}{2} \times c_g. \tag{9}$$

Since $\frac{1}{2}g\rho a^2$ is the whole energy per unit length, (9) shows that the energy is transmitted at a rate equal to the group velocity.

Ex.1. When simple harmonic waves of length λ are propagated over the surface of deep water, prove that, at a point whose depth below the undisturbed surface is h, the pressure at the instants when the disturbed depth of the point is $h+\eta$ bears to the undisturbed pressure at the same point the ratio

$$\left(1+\frac{\eta}{h}e^{-2\pi h/\lambda}\right) : 1,$$

atmospheric pressure and surface tension being neglected.

Solution. For deep water, the velocity potential is given by

$$\phi = \left(\frac{na}{m}\right) e^{my} \cos(mx - nt) \tag{1}$$

$$\therefore \quad \frac{\partial \phi}{\partial t} = \left(\frac{an^2}{m}\right) e^{my} \sin(mx - nt) \tag{2}$$

Also $\eta = a\sin(mx - nt)$ and

$$c^2 = \frac{n^2}{m^2} = \frac{g}{m}$$

So (2) becomes

$$\frac{\partial \phi}{\partial t} = g\eta e^{my} \tag{3}$$

Pressure at any point within the water is given by

$$\frac{p}{\rho} - \left(\frac{\partial \phi}{\partial t}\right) + gy = c \quad \text{(a constant)} \tag{4}$$

When $y = 0$, $p = 0$, and $\dfrac{\partial \phi}{\partial t} = 0$, so $c = 0$ and hence (4) gives

$$\therefore \quad p = \rho\left(\frac{\partial \phi}{\partial t}\right) - \rho g y$$

or

$$p = \rho g \eta e^{my} - \rho g y, \quad \text{by (3)} \tag{5}$$

Disturbed pressure p_1 when $y = -h$ is given by

$$p_1 = \rho g \eta e^{-mh} + \rho g h = \rho g h \left\{ 1 + \frac{\eta}{h} e^{-mh} \right\} \tag{6}$$

Undisturbed pressure p_2 at a depth h is given by

$$p_2 = \rho g h \tag{7}$$

$$\therefore \quad p_1 : p_2 = \left(1 + \frac{\eta}{h} e^{-mh} \right) : 1$$

or,

$$p_1 : p_2 = \left(1 + \frac{\eta}{h} e^{-2\pi h/\lambda} \right) : 1, \quad \text{as } m = \frac{2\pi}{\lambda}$$

Ex.2. Two dimensional waves of length $2\pi/m$ are produced at the surface of separation of two liquids which are of densities ρ, ρ' $(\rho' > \rho)$ and depths h, h' confined between two fixed horizontal planes. Prove that, if the potential energy is reckoned zero in the position of equilibrium, the total energy of the lower liquid to that of the upper is in the ratio

$$\rho\{(2\rho - \rho')\coth mh + \rho'\coth mh'\} : \rho'\{(\rho - 2\rho')\coth mh' - \rho\coth mh\}$$

Solution. Let the wave profile be of the form

$$\eta = a\sin(mx - nt), \tag{1}$$

for lower liquid

$$\phi = \frac{ac}{\sinh mh}\cosh m(y+h)\cos(mx - nt), \tag{2}$$

and for upper liquid

$$\phi' = -\frac{ac}{\sinh mh'} \cosh m(y+h') \cos(mx - nt). \tag{3}$$

where

$$g(\rho - \rho') = c^2 m(\rho \coth mh + \rho' \coth mh'). \tag{4}$$

Kinetic energy of the lower liquid per wave length

$$T_1 = \frac{1}{2}\rho \int_0^\lambda \eta \left(\phi \frac{\partial \phi}{\partial y}\right)_{y=0} dx = \frac{1}{2}\rho a^2 c^2 m \coth mh \int_0^\lambda \cos^2(mx - nt)\, dx$$

Thus,

$$T_1 = \frac{1}{2}\rho a^2 c^2 m \coth mh \cdot \frac{\lambda}{2} = \frac{1}{4}\lambda \rho a^2 c^2 m \coth mh.$$

Potential energy of the lower liquid per wave length

$$V_1 = \frac{1}{2} g\rho \int_0^{\lambda} \eta^2 dx = \frac{1}{2} g\rho a^2 \int_0^{\lambda} \sin^2(mx - nt)\, dx = \frac{1}{2} g\rho a^2 \frac{\lambda}{2}.$$

So the total energy of the lower liquid is given by

$$E_1 = T_1 + V_1 = \frac{1}{4} g a^2 \rho\lambda \left[1 + \frac{c^2 m \coth mh}{g}\right]$$

$$= \frac{1}{4} g a^2 \rho\lambda \left[1 + \frac{(\rho - \rho') \coth mh}{\rho \coth mh + \rho' \coth mh'}\right], \quad \text{by (4)}$$

Thus,

$$E_1 = \frac{1}{4} g a^2 \rho\lambda \frac{(2\rho - \rho') \coth mh + \rho' \coth mh'}{\rho \coth mh + \rho' \coth mh'}. \tag{5}$$

Similarly the kinetic and potential energies T_2 and V_2 for the upper fluid are given by

$$T_2 = \frac{1}{4}\lambda\rho' a^2 c^2 m \coth mh', \qquad V_2 = -\frac{1}{4} g\rho' a^2 \lambda.$$

So the total energy of the upper fluid is given by

$$E_2 = T_2 + V_2 = \frac{1}{4} g a^2 \rho' \lambda \left[\frac{(\rho - \rho') \coth mh'}{\rho \coth mh + \rho' \coth mh'} - 1 \right]$$

$$= \frac{1}{4} g a^2 \rho' \lambda \frac{(\rho - 2\rho') \coth mh' - \rho \coth mh}{\rho \coth mh + \rho' \coth mh'}. \qquad (6)$$

From (5) and (6) we get,

$$\rho\{(2\rho - \rho') \coth mh + \rho' \coth mh'\} : \rho'\{(\rho - 2\rho') \coth mh' - \rho \coth mh\}$$

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