Waves - Fluid Dynamics II

Presented by

MATHEMATICAL EXPLORATIONS

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Mathematical representation of wave motion:

Let the x-axis be taken horizontally and the y-axis vertically upward.
Let us consider the motion of a free surface (such as the surface of water) and at any time t, the vertical displacement of a particle of the surface from its mean position is denoted by y.
Let the equation of the free surface at time t is

$$y = a \sin(mx - nt), \tag{1}$$

where,

- a = amplitude of the wave (maximum displacement).
- m = wave number (related to wavelength).
- n = angular frequency (related to time period).

This equation represents a *simple harmonic progressive wave*.

Interpretation as a Moving Wave:

Let us consider the wave profile at time $t = 0$:

$$y = a\sin(mx)$$

Now suppose we increase time by T and simultaneously increase x by $\left(\frac{n}{m}\right)T$. Then,

$$\text{R.H.S. of (2)} = a\sin m\left(x + \frac{n}{m}T - \frac{n}{m}(t+T)\right)$$

$$= a\sin m\left(x - \frac{n}{m}t\right) = \text{L.H.S. of (1)}$$

Thus, the wave retains exactly the same shape if x increases by $\frac{n}{m}T$ when time increases by T. This proves that the wave form moves forward without changing shape.

Velocity of Propagation:

Since the wave advances a distance $\frac{n}{m}T$ in time T, the speed is

$$c = \frac{n}{m}.$$

This constant c is called the *velocity of propagation* of the wave. Because the expression is $(mx - nt)$, the wave moves in the **positive x-direction**.

Mean Level and Amplitude:

If $a = 0$, then

$$y = 0,$$

which represents the undisturbed surface. This is called the *mean level*. The maximum value of y is a, hence a is called the *amplitude* of the wave.

Crests and Troughs:

- Points of maximum elevation are called *crests*.
- Points of maximum depression are called *troughs*.

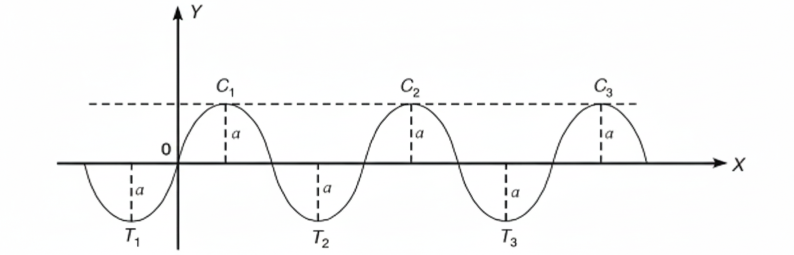


Figure 1: Wave profile showing crests and troughs

Wavelength:

The distance between two successive crests (or troughs) is called the *wavelength* and is denoted by λ. Since sine repeats after an angle 2π, $mx = 2\pi \Rightarrow \lambda = \frac{2\pi}{m}$.

Time Period:

The wave repeats its motion when $nt = 2\pi$. Thus, $T = \frac{2\pi}{n}$. T is called the *period* of the wave.

Relation Between Speed, Wavelength and Period:

We have

$$c = \frac{n}{m}, \quad \lambda = \frac{2\pi}{m}, \quad T = \frac{2\pi}{n}.$$

Eliminating m and n, we get

$$T = \frac{\lambda}{c} \Rightarrow c = \frac{\lambda}{T}$$

Frequency:

The reciprocal of the period is called the *frequency*:

$$f = \frac{1}{T}.$$

It represents the number of oscillations per second.

Phase and Phase Angle:

The quantity $mx - nt$ is called the *phase angle*. If the equation of wave motion is $y = a\sin(mx - nt + \varepsilon)$ then ε is called the *phase constant* or *initial phase*. It determines the starting position of the wave.

Important Points to Remember:

- Each particle performs simple harmonic motion.
- The wave travels with constant speed $c = \frac{n}{m}$.
- The wave shape does not change during propagation.
- λ measures spatial repetition.
- T measures temporal repetition.
- f measures oscillations per second.

Standing or Stationary Waves:

Let us consider two simple harmonic progressive waves having:

- Same amplitude,
- Same wavelength,
- Same period,
- Traveling in opposite directions.

Their equations are

$$y_1 = \frac{a}{2}\sin(mx - nt) \quad \text{(wave moving in +x direction)}$$

$$y_2 = \frac{a}{2}\sin(mx + nt) \quad \text{(wave moving in -x direction)}$$

Since the motion is linear and small, we apply the principle of superposition

$$y = y_1 + y_2$$

Substituting,

$$y = \frac{a}{2}\sin(mx - nt) + \frac{a}{2}\sin(mx + nt).$$

Now,

$$\sin(A - B) + \sin(A + B) = 2\sin A\cos B.$$

Let,

$$A = mx, \quad B = nt.$$

Then,

$$y = \frac{a}{2}\cdot 2\sin mx\cos nt = a\sin mx\cos nt$$

This is the equation of a **standing (stationary) wave**.

Why Standing Wave?

In the equation, $y = a \sin mx \cos nt$

- $\sin mx$ depends only on position.
- $\cos nt$ depends only on time.

Thus,

- The spatial shape $\sin mx$ does not travel.
- Only the amplitude at each point changes with time.

Hence, the wave does not move forward — it simply oscillates in place.

Nature of Motion at a Fixed Point

At a fixed position x:

$$y = (a \sin mx) \cos nt.$$

So each particle performs simple harmonic motion with amplitude $A(x) = a \sin mx$. Thus, the amplitude varies with position.

Nodes

Nodes occur where the displacement is always zero. That means $\sin mx = 0$. So, $mx = n\pi \Rightarrow x = \frac{n\pi}{m}$.
Since wavelength $\lambda = \frac{2\pi}{m}$,

$$x = \frac{n\lambda}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

These points are called *nodes*. At nodes, particles never move.

Antinodes

Antinodes occur where amplitude is maximum. Maximum amplitude occurs when

$$|\sin mx| = 1.$$

Thus,

$$mx = \frac{(2n+1)\pi}{2}$$

$$x = \frac{(2n+1)\lambda}{4}.$$

These points are called *antinodes*. At antinodes, particles oscillate with maximum amplitude a.

Key Characteristics of Standing Waves

- No energy is transported along the medium.
- Nodes remain fixed in position.
- Amplitude varies from point to point.
- Distance between consecutive nodes $= \lambda/2$.
- Distance between node and adjacent antinode $= \lambda/4$.

Progressive Wave as Combination of Standing Waves:

Now let us consider two stationary waves

$$y_3 = a \sin mx \cos nt, \quad y_4 = a \cos mx \sin nt$$

Adding we get,

$$y = y_3 \pm y_4$$

Using identity

$$\sin A \cos B \pm \cos A \sin B = \sin(A \pm B)$$

we get,

$$y = a \sin(mx \pm nt)$$

This is the equation of a *progressive wave*.

A progressive wave can be regarded as the superposition of two stationary waves

- Same amplitude, Same wavelength, Same Period
- Phase difference of $\pi/2$ (quarter period)

Thus, standing waves arise from two progressive waves and a progressive wave can be resolved into two standing waves.

- Progressive wave $\rightarrow$ Energy travels.
- Standing wave $\rightarrow$ Energy trapped locally.
- Nodes $\rightarrow$ zero displacement.
- Antinodes $\rightarrow$ maximum displacement.

Types of Liquid Waves:

Liquid waves may be divided into the following two classes:

(i) **Long waves in shallow water or tidal waves.**
Such waves arise when the depth of the liquid is small compared to the wavelength and the disturbance affects the motion of the whole of the liquid. In these waves the vertical acceleration of the liquid is negligible as compared with the horizontal acceleration and the plane of the liquid moves as a whole.

(ii) **Surface waves.**
Such waves occur when the wavelength of the oscillations is small compared to the depth of the liquid and hence the disturbance does not extend far below the surface. In these waves the vertical acceleration is appreciable and so it cannot be neglected. Wind waves and surface tension waves are examples of surface waves. Such waves occur in deep and unbounded (in horizontal directions) liquids like lakes and oceans.

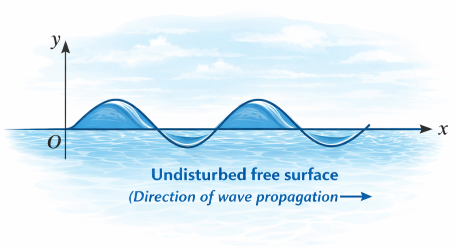
Surface Waves:

Surface waves occur at and near the free surface of an unbounded sheet of liquid where the depth is large compared to the wavelength. Since the vertical acceleration is comparable with the horizontal acceleration, forces in both directions must be considered.

Let the x–axis be taken along the undisturbed free surface in the direction of propagation of the waves, and the y–axis vertically upwards.

We assume that the motion is irrotational, incompressible and two-dimensional.

Since the flow is irrotational, we have



$$\nabla \times \vec{q} = 0$$

This implies that a velocity potential $\phi(x, y, t)$ exists such that

$$\vec{q} = -\nabla\phi$$

Hence, the velocity components become

$$u = -\frac{\partial\phi}{\partial x}, \qquad v = -\frac{\partial\phi}{\partial y} \tag{1}$$

Since the liquid is incompressible, the continuity equation gives

$$\nabla \cdot \vec{q} = 0$$

Substituting $\vec{q} = -\nabla\phi$, we obtain

$$\begin{aligned} &\nabla \cdot (-\nabla\phi) = 0 \\ &\Rightarrow -\nabla^2\phi = 0 \\ &\Rightarrow \nabla^2\phi = 0 \end{aligned}$$

In two dimensions, this becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2}$$

At a fixed vertical wall, fluid cannot penetrate it. Hence the normal velocity must vanish. If the wall is vertical (normal in the x-direction), then $u = 0$.
Since $u = -\frac{\partial\phi}{\partial x}$, so we get

$$\frac{\partial\phi}{\partial x} = 0 \tag{3}$$

For unsteady, irrotational flow, Bernoulli's equation is

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \frac{q^2}{2} - gy = F(t), \tag{4}$$

where $q^2 = u^2 + v^2$.
The terms represent,

- $\dfrac{\partial \phi}{\partial t}$: unsteady acceleration,
- $-\dfrac{q^2}{2}$: kinetic energy per unit mass,
- $-gy$: gravitational potential energy,
- $F(t)$: arbitrary function of time.

The free surface is exposed to the atmosphere, hence $p = \text{constant}$.
Therefore, along a particle on the free surface,

$$\frac{Dp}{Dt} = 0 \tag{5}$$

The material derivative is given by

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}$$

Substituting u and v from equation (1), we get

$$\frac{\partial p}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial p}{\partial y} = 0 \tag{6}$$

For small wave motion,

- velocities are small,

- squares of small quantities are negligible.

Hence we neglect $\dfrac{q^2}{2}$ in equation (4).
Also, since $F(t)$ is arbitrary, we may absorb it into ϕ and take

$$F(t) = 0$$

Thus equation (4) reduces to

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy \Rightarrow p = \rho\left(\frac{\partial \phi}{\partial t} - gy\right) \tag{7}$$

Now,

$$\frac{\partial p}{\partial t} = \rho \frac{\partial^2 \phi}{\partial t^2},$$

$$\frac{\partial p}{\partial x} = \rho \frac{\partial^2 \phi}{\partial x \partial t},$$

$$\frac{\partial p}{\partial y} = \rho \left(\frac{\partial^2 \phi}{\partial y \partial t} - g \right)$$

Substituting into equation (6), we obtain

$$\rho \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial x} \left(\rho \frac{\partial^2 \phi}{\partial x \partial t} \right) - \frac{\partial \phi}{\partial y} \left[\rho \left(\frac{\partial^2 \phi}{\partial y \partial t} - g \right) \right] = 0$$

Dividing throughout by ρ, we get

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x \partial t} - \frac{\partial \phi}{\partial y} \left(\frac{\partial^2 \phi}{\partial y \partial t} - g \right) = 0 \tag{8}$$

or, omitting the second and third terms which are of the same order as q^2, we get

$$\frac{\partial^2 \phi}{\partial t^2} + g\left(\frac{\partial \phi}{\partial y}\right) = 0 \tag{9}$$

Condition (1) must be satisfied at the free surface.
If η is the elevation of the free surface at time t above the point whose abscissa is x, the equation of the free surface is given by

$$\eta = f(x,t) \qquad \text{or} \qquad \eta - f(x,t) = 0$$

But we know that if $F(x,\eta,t) = \eta - f(x,t) = 0$ be the boundary surface, then we must have

$$\frac{DF}{Dt} = 0 \quad \Rightarrow \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial \eta} = 0 \quad \Rightarrow \qquad -\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial x} + v = 0 \tag{10}$$

Now $\partial f/\partial t$ is $\dot{\eta}$. Again $\partial f/\partial x$ or $\partial \eta/\partial x$ being the tangent of the slope of the free surface is small so that the second term in (10) can be omitted. Then (10) reduces to

$$\dot{\eta} = v = -\frac{\partial \phi}{\partial y}, \tag{11}$$

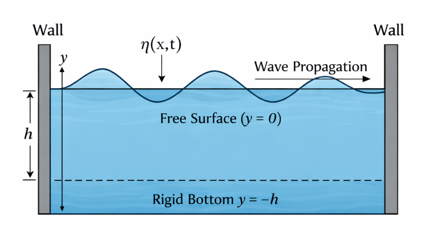
which holds at the free surface.

Thus for the surface waves the velocity potential is a solution of Laplace's equation (1) which makes $\partial \phi/\partial x = 0$ *as a fixed boundary and satisfies (9) and (11) at the free surface of the liquid.*

Case I. Progressive waves on the surface of a canal:

We consider progressive surface waves in a canal of:

- Uniform depth $= h$
- Rigid bottom at $y = -h$
- Free surface at $y = 0$
- Wave travelling in the positive x-direction



The surface elevation is given as

$$\eta = a \sin(mx - nt) \tag{13}$$

where, a = amplitude, m = wave number, n = angular frequency, $c = \frac{n}{m}$ (wave speed) and $\lambda = \frac{2\pi}{m}$ (wavelength).
For incompressible, irrotational flow,

$$\nabla^2\phi = 0 \quad \text{(Laplace's equation)} \tag{1}$$

Boundary Conditions:

(A) Bottom condition (Rigid boundary)

At the bottom $y = -h$,

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = -h \tag{14}$$

(B) Dynamic free surface condition

At the free surface $y = 0$,

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \tag{15}$$

(C) Kinematic free surface condition

$$v = \frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} \quad \text{at} \quad y = 0 \tag{16}$$

Given,

$$\eta = a \sin(mx - nt)$$

Differentiating with respect to time,

$$\frac{\partial \eta}{\partial t} = -an \cos(mx - nt)$$

Using (16),

$$-\frac{\partial \phi}{\partial y} = -an \cos(mx - nt)$$

Hence,

$$\frac{\partial \phi}{\partial y} = an \cos(mx - nt) \quad \text{at} \quad y = 0 \tag{17}$$

Since the boundary condition contains $\cos(mx - nt)$, let us assume

$$\phi = f(y) \cos(mx - nt) \tag{18}$$

Laplace equation is given by

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Now,

$$\frac{\partial^2}{\partial x^2} [\cos(mx - nt)] = -m^2 \cos(mx - nt)$$

Substituting,

$$f''(y)\cos(mx-nt)-m^2f(y)\cos(mx-nt)=0$$

Hence,

$$f''(y)-m^2f(y)=0 \tag{19}$$

Characteristic equation is

$$r^2-m^2=0$$

$$r=\pm m$$

Thus,

$$f(y)=Ae^{my}+Be^{-my}$$

Therefore,

$$\phi = (Ae^{my} + Be^{-my})\cos(mx - nt) \tag{20}$$

Differentiating we get,

$$\frac{\partial \phi}{\partial y} = m(Ae^{my} - Be^{-my})\cos(mx - nt)$$

At $y = -h$,

$$m(Ae^{-mh} - Be^{mh}) = 0$$

Hence,

$$Ae^{-mh} = Be^{mh}$$

Let,

$$Ae^{-mh} = Be^{mh} = \frac{D}{2}$$

Then,

$$\phi = D\cosh m(y+h)\cos(mx-nt) \tag{21}$$

Using (15),

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0$$

$$\frac{\partial^2}{\partial t^2}[\cos(mx-nt)] = -n^2\cos(mx-nt)$$

$$\frac{\partial\phi}{\partial y} = Dm\sinh m(y+h)\cos(mx-nt)$$

At $y = 0$,

$$-n^2 D \cosh(mh) \cos(...) + gDm \sinh(mh) \cos(...) = 0$$

Canceling common terms,

$$-n^2 \cosh(mh) + gm \sinh(mh) = 0$$

Thus,

$$n^2 = gm \tanh(mh) \tag{22}$$

Let,

$$c = \frac{n}{m}, \quad \lambda = \frac{2\pi}{m} \tag{23}$$

Then,

$$c^2 = \frac{n^2}{m^2} = \frac{g}{m}\tanh(mh) \tag{24}$$

or

$$c^2 = \left(\frac{g\lambda}{2\pi}\right)\tanh\left(\frac{2\pi h}{\lambda}\right) \tag{25}$$

We now determine the constant D of (21) in terms of the amplitude a of the wave. Using (13) and (21), the boundary condition (16) gives

$$-na = mD\sinh mh$$

so that

$$\phi = \frac{na}{m}\frac{\cosh m(y+h)}{\sinh mh}\cos(mx - nt) \tag{26}$$

or using (22)

$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt) \tag{27}$$

From (26) the velocity components of the particles are

$$u = -\frac{\partial \phi}{\partial x} = na\frac{\cosh m(y+h)}{\sinh mh}\sin(mx - nt) \tag{28}$$

$$v = -\frac{\partial \phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh}\cos(mx - nt) \tag{29}$$

We now determine the path of the particles. Let (x', y') be the coordinates of a particle relative to its mean position (x, y) such that

$$|x'| = |x + y'| \text{ is very small.}$$

Neglecting the squares of small quantities, for a wave of small elevation the velocities at $z = x + iy$ and $z + z' = (x + x') + i(y + y')$ will be equal. Hence we may write

$$\frac{dx'}{dt} = u = na\frac{\cosh m(y+h)}{\sinh mh}\sin(mx - nt) \tag{30}$$

and

$$\frac{dy'}{dt} = v = -na\frac{\sinh m(y+h)}{\sinh mh}\cos(mx-nt) \tag{31}$$

Integrating w.r.t. t, (30) and (31) give

$$x' = a\frac{\cosh m(y+h)}{\sinh mh}\cos(mx-nt) \tag{32}$$

$$y' = a\frac{\sinh m(y+h)}{\sinh mh}\sin(mx-nt) \tag{33}$$

Let

$$a' = a\frac{\cosh m(y+h)}{\sinh mh}, \qquad b' = a\frac{\sinh m(y+h)}{\sinh mh} \tag{34}$$

Using (34) and eliminating t from (32) and (33), we get

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1 \tag{35}$$

This shows that each water particle moves in an **elliptical path** about its mean (equilibrium) position.

From,

$$x' = a' \cos(mx - nt), \qquad y' = b' \sin(mx - nt),$$

the quantity $(mx - nt)$ plays the role of the *eccentric angle* of the ellipse. Since n is constant, $(mx - nt)$ increases uniformly with time. Therefore, the particle moves around the ellipse with constant angular speed.

For an ellipse, the distance between the foci is

$$2\sqrt{a'^2 - b'^2}.$$

Using the identity $\cosh^2 u - \sinh^2 u = 1$, this becomes $2a \csc(mh)$. This is a constant (independent of depth), so all such ellipses have the same focal distance.

The semi-axes are

$$a' = a\frac{\cosh m(y+h)}{\sinh mh}, \qquad b' = a\frac{\sinh m(y+h)}{\sinh mh}.$$

As the depth increases (i.e., as y decreases), both a' and b' decrease. Thus, the size of the ellipse becomes smaller with depth.
At the bottom, where $y = -h$,

$$b' = 0,$$

so the vertical motion vanishes.
Hence, the ellipse degenerates into a straight horizontal line, and the particles execute simple to-and-fro horizontal motion.

Thus,

- Near the surface: motion is nearly circular (in deep water).
- At intermediate depths: motion is elliptical.
- At the bottom: motion is purely horizontal.

Case II. Progressive waves on a deep canal:

If the depth h of the canal is very large compared with the wavelength λ, then e^{-mh} is negligible.

In Case I, the general solution was

$$\phi = (Ae^{my} + Be^{-my}) \cos(mx - nt).$$

Since $e^{-mh} \approx 0$ for large h, boundedness of the motion as $y \to -\infty$ requires

$$B = 0.$$

Hence the velocity potential reduces to

$$\phi = Ae^{my} \cos(mx - nt). \tag{20'}$$

From the free-surface boundary condition, we obtain

$$n^2 = \frac{g}{m}, \tag{22'}$$

which gives the phase velocity

$$c = \frac{n}{m}.$$

Since $m = \frac{2\pi}{\lambda}$, we get

$$c^2 = \frac{g\lambda}{2\pi}. \tag{25'}$$

Thus, in deep water the wave velocity is independent of depth.

Let the surface elevation be

$$\eta = a\cos(mx - nt).$$

Using the kinematic boundary condition and substituting into (20'), we obtain

$$na = mA.$$

Therefore,

$$A = \frac{na}{m}.$$

Substituting into (20'),

$$\phi = \left(\frac{na}{m}\right) e^{my} \cos(mx - nt). \tag{26'}$$

Using $n^2 = \frac{g}{m}$, this may also be written as

$$\phi = \left(\frac{ga}{n}\right) e^{my} \cos(mx - nt). \tag{27'}$$

The velocity components are

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}.$$

Hence,

$$u = na\, e^{my} \sin(mx - nt), \tag{28'}$$

$$v = -na\, e^{my} \cos(mx - nt). \tag{29'}$$

Let (x', y') be the displacement of a particle from its mean position (x, y). Then

$$\frac{dx'}{dt} = u, \qquad \frac{dy'}{dt} = v.$$

Integrating with respect to t, we obtain

$$x' = ae^{my} \cos(mx - nt), \qquad y' = ae^{my} \sin(mx - nt).$$

Squaring and adding,

$$x'^2 + y'^2 = (ae^{my})^2.$$

Thus, each particle moves in a **circle** of radius

$$R = ae^{my}.$$

Since $y < 0$ below the surface, e^{my} decreases with depth, so the radius decreases exponentially with depth.
At the free surface ($y = 0$),

$$R = a,$$

i.e., the radius equals the wave amplitude.
The motion is uniform circular motion with angular velocity n.

Remark 1: Comparison with finite depth

In finite depth, the velocity differs by a factor

$$\sqrt{\tanh\left(\frac{2\pi h}{\lambda}\right)}.$$

If $h > \frac{\lambda}{2}$, then $\tanh(2\pi h/\lambda) \approx 1$, and the deep-water approximation is valid.

Remark 2: Complex potential

Since $nm = c$, equation (26') becomes

$$\phi = ac\, e^{my} \cos(mx - nt).$$

Using the relation

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x},$$

we obtain

$$\frac{\partial \psi}{\partial y} = -acme^{my} \sin(mx - nt).$$

Integrating with respect to y,

$$\psi = -ace^{my}\sin(mx - nt).$$

Hence the complex potential

$$w = \phi + i\psi$$

is

$$w = ace^{my}\left[\cos(mx - nt) - i\sin(mx - nt)\right].$$

Using Euler's formula,

$$w = ace^{my}e^{-i(mx-nt)} = ace^{imz-int}.$$

Therefore,

$$\boxed{w = ac\,e^{i(mz-nt)}},$$

which is the required complex potential for a progressive wave in deep water.

Case III. Stationary waves on the surface of a canal:

Consider a stationary wave of the type

$$\eta = a \sin mx \cos nt. \tag{13A}$$

at the surface of canal of uniform depth h and having parallel vertical walls. Let the free surface be along the x-axis (i.e. $y = 0$) so that equation of the bottom (rigid boundary) is $y = -h$. Then we must find ϕ satisfying (1) and subjected to the following boundary conditions

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = -h \tag{14A}$$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{15A}$$

$$v = \frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y} = \quad \text{at} \quad y = 0 \tag{16A}$$

Using (13A), (16A) gives

$$\frac{\partial \phi}{\partial y} = an \sin mx \sin nt \quad \text{at} \quad y = 0 \tag{17A}$$

Equation (17A) suggests that we should take the solution of (1) of the form

$$\phi = f(y) \sin mx \sin nt \tag{18A}$$

Substituting this in (1), we obtain

$$\frac{d^2 f}{dy^2} - m^2 f = 0 \tag{19A}$$

whose solution is

$$f(y) = Ae^{my} + Be^{-my}, \quad A, B \text{ being arbitrary constants.}$$

and hence

$$\phi = (Ae^{my} + Be^{-my}) \sin mx \sin nt \tag{20A}$$

Using (14A), (20A) gives

$$Ae^{-mh} = Be^{mh} = \frac{D}{2}, \text{ say,}$$

so that

$$\phi = D \cosh m(y+h) \sin mx \sin nt \tag{21A}$$

Again, using (15A), gives

$$n^2 = gm \tanh mh \tag{22A}$$

Let

$$c = \frac{n}{m}, \qquad \lambda = \frac{2\pi}{m} \tag{23A}$$

denote the velocity of propagation and the wave length respectively.

Then (22A) reduces to

$$c^2 = \frac{g}{m} \tanh mh \tag{24A}$$

or

$$c^2 = \left(\frac{g\lambda}{2\pi}\right) \tanh\left(\frac{2\pi h}{\lambda}\right) \tag{25A}$$

We now determine the constant D of (21A) in terms of the amplitude a of the wave. Using (13A) and (21A), the boundary condition (16A) gives

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and

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To determine the path of the particles in stationary waves.

With the same notations and method as in Case I, we have

$$\frac{dx'}{dt} = -\frac{\partial \phi}{\partial x} = -na \frac{\cosh m(y+h)}{\sinh mh} \cos mx \sin nt \tag{30A}$$

and

$$\frac{dy'}{dt} = -\frac{\partial \phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh} \sin mx \sin nt \tag{31A}$$

Integrating w.r.t. t', (30A) and (31A) give

$$x' = a\frac{\cosh m(y+h)}{\sinh mh} \cos mx \cos nt \tag{32A}$$

and

$$y' = a\frac{\sinh m(y+h)}{\sinh mh} \sin mx \sin nt \tag{33A}$$

Hence

$$\frac{y'}{x'} = \tan m(y+h) \tan mx, \tag{34A}$$

and since this is independent of t the motion of each particle is rectilinear. The direction of motion varies from vertical below the crests and troughs [where $mx = (n + 1/2)\pi$], to horizontal below the nodes (where $mx = n\pi$).

Case IV. Stationary waves on a deep canal:

If the depth h of the canal is sufficiently great in comparison with λ for e^{-mh} to be neglected, then in case III we must have $B = 0$. Thus, we have, instead of (20A),

$$\phi = Ae^{my} \sin mx \sin nt, \tag{20A$'$}$$

instead of (22A)

$$n^2 = gm, \tag{22A$'$}$$

and instead of (25A)

$$c^2 = \frac{g\lambda}{2\pi}. \tag{25A$'$}$$

We now determine the constant A of (20A$'$) in terms of the amplitude of the wave. Using (13A) and (20A$'$), the boundary condition (16A) gives $na = mA$, so that

$$\phi = \left(\frac{na}{m}\right) e^{my} \sin mx \sin nt, \tag{26A$'$}$$

or

$$\phi = \left(\frac{ga}{n}\right) e^{my} \sin mx \sin nt. \tag{27A$'$}$$

The velocity components of the particles are

$$u = -\left(\frac{\partial \phi}{\partial x}\right) = -nae^{my} \cos mx \sin nt, \tag{28A$'$}$$

$$v = -\left(\frac{\partial \phi}{\partial y}\right) = -nae^{my} \sin mx \sin nt. \tag{29A$'$}$$

Following the procedure of Case III we obtain in this case

$$x' = ae^{my} \cos mx \cos nt,$$

and

$$y' = ae^{my}\sin mx \cos nt.$$

Hence

$$\frac{y'}{x'} = \tan mx,$$

showing that the path of the particle is a straight line. The amplitude of oscillations is ae^{my} which decreases as depth increases. Moreover the particles oscillate in the vertical direction at the antinodes, whereas they oscillate in the horizontal direction at the nodes.

THANK YOU

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