Intrinsic Properties of a Surface
Differential Geometry

Presented by

MATHEMATICAL EXPLORATIONS

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Surface:

A surface is said to be the locus of a point whose cartesian coordinates (x, y, z) are functions of two independent parameters, say (u, v), i.e.,

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v). \tag{1}$$

Above equations are called *parametric* or *freedom equations* of a surface, where the parameters u, v assume real values and vary in some region, say D.

We may define a surface vectorially as the locus of a point whose position vector $\vec{r}$ is expressed in terms of parameters, i.e.,

$$\vec{r} = \vec{r}(u, v)$$

represents a surface.

The above representation of the surface is due to Gauss and is termed as the *Gaussian form* of the surface. The parameters u and v are called *curvilinear coordinates* or *surface co-ordinates* of the current point on the surface. The pair of parameters u and v represents the point (u, v).

Remark.

- The representation of surface given by equation (1) is called an *explicit representation* of a surface.

Class of a Surface:

If the equation of the surface be

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v),$$

then the surface is said to be of class r if the functions f, g, h are single valued as well continuous and possess partial derivatives of r^{th} order.

Remark.

We represent the partial differentiations with respect to the parameters u and v by the suffixes 1 and 2 respectively, i.e.,

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v}, \quad \mathbf{r}_{11} = \frac{\partial^2 \mathbf{r}}{\partial u^2}, \quad \mathbf{r}_{12} = \frac{\partial^2 \mathbf{r}}{\partial u \partial v}, \quad \mathbf{r}_{22} = \frac{\partial^2 \mathbf{r}}{\partial v^2}.$$

Regular (or Ordinary) Point and Singularities on a Surface:

Let us consider a point P on the surface whose position vector is $\mathbf{r}$. Then

$$\mathbf{r} = (x, y, z) = \big(x(u,v),\ y(u,v),\ z(u,v)\big).$$

$$\therefore \quad \mathbf{r}_1 = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad \mathbf{r}_2 = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

Now the point P is called a *regular point* or *ordinary point* if

$$\mathbf{r}_1 \times \mathbf{r}_2 \neq 0,$$

i.e., if the rank of the matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

is 2 (i.e., at least one of the second order minor does not vanish).

In case $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ at a point P on the surface, then the point P is called a *singularity* of the surface. We have the following two types of singularities.

1. **Essential Singularities.** These singularities are due to the nature (geometrical features) of the surface and are independent of the choice of parametric representation of the surface, e.g., the vertex of the cone is an essential singularity.

2. **Artificial Singularities.** These singularities are due to a particular choice of parametric representation. For example, the pole in the plane referred to polar coordinates is an artificial singularity. Let $\mathbf{r}$ be the position vector of any point in the plane. When referred to polar coordinates (r, θ), we have

   $$\mathbf{r} = (r\cos\theta, r\sin\theta, 0).$$

   If $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the partial derivatives with respect to r and θ respectively, then

   $$\mathbf{r}_1 = (\cos\theta, \sin\theta, 0), \quad \mathbf{r}_2 = (-r\sin\theta, r\cos\theta, 0).$$

   $$\therefore \quad \mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\mathbf{k}.$$

Thus,

$$\mathbf{r}_1 \times \mathbf{r}_2 = 0 \quad \text{when } r = 0, \text{ i.e., at the pole.}$$

Now consider the Cartesian system, where $\mathbf{r} = (x, y, 0)$. Then

$$\mathbf{r}_1 = (1, 0, 0), \quad \mathbf{r}_2 = (0, 1, 0)$$

$$\Rightarrow \mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{k} \neq 0 \quad \text{at the pole.}$$

Thus there is an artificial singularity at $r = 0$.

Tangent Plane and Normal:

To find the equation of the tangent plane to a surface at a given point $P(\mathbf{r})$.

$$\frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{ds} \quad \text{or} \quad \vec{t} = \vec{r}_1\frac{du}{ds} + \vec{r}_2\frac{dv}{ds}.$$

Now for the curve $u = c_1$ (constant), $\vec{r}_1 = 0$, so that the unit vector along the tangent is parallel to $\vec{r}_2$. Similarly, the unit vector along the tangent to the curve $v = c_2$ (constant) is parallel to $\vec{r}_1$.

The tangent plane at P contains $\vec{r}_1$ and $\vec{r}_2$ at P. Then the equation of the tangent plane is

$$\vec{R} = \vec{r} + c_1\vec{r}_1 + c_2\vec{r}_2,$$

where $\vec{R}$ is the position vector of a current point on the plane.

The equation of the tangent plane can be expressed as

$$(\vec{R} - \vec{r}) \cdot (\vec{r}_1 \times \vec{r}_2) = 0.$$

Since $\vec{r}_1 \times \vec{r}_2$ gives the direction of the normal to the tangent plane, it is perpendicular to any line $\vec{R} - \vec{r}$ in the plane.

Normal Line. The normal to the tangent at P is the line passing through $P(\vec{r})$ and parallel to the vector $\vec{r}_1 \times \vec{r}_2$. Hence, the equation of the normal line at P to the surface is given by

$$\vec{R} = \vec{r} + \lambda(\vec{r}_1 \times \vec{r}_2).$$

The normal to the surface at P is the same as the normal to the tangent plane at P, and therefore the *unit* surface normal $\hat{N}$ is given by

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H},$$

where $H = |\vec{r}_1 \times \vec{r}_2| \neq 0$.
The direction of $\hat{N}$ is fixed by convention such that $\vec{r}_1 \times \vec{r}_2$ and $\hat{N}$ form a right-handed system.

To Find the Cartesian Equation of the Tangent Plane to the Surface $F(x, y, z) = 0$:

We know that

$$\nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

is normal to the tangent plane at $P(x, y, z)$. Thus, it is perpendicular to any line in the plane.

Let $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be any point on the plane, then

$$(\vec{R} - \vec{r}) \cdot \nabla F = 0.$$

$$[(X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}] \cdot \left(\frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}\right) = 0.$$

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0 \tag{1}$$

which is the equation of the tangent plane at P.

Again x, y, z are functions of u and v, thus

$$\frac{dF}{du} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u} = 0. \tag{2}$$

and

$$\frac{dF}{dv} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v} = 0. \tag{3}$$

On solving (2) and (3), we get

$$\frac{\frac{\partial F}{\partial x}}{\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}} = \frac{\frac{\partial F}{\partial y}}{\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}} = \frac{\frac{\partial F}{\partial z}}{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}} = k.$$

Therefore, the equation of the tangent plane can be put in the form

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0$$

To Find the Cartesian Equation of the Normal to the Surface $F(x, y, z) = 0$ at $P(x, y, z)$:

Let $\vec{R}$ be the position vector of any point $Q(X, Y, Z)$ on the normal and $\vec{r}$ be the position vector of $P(x, y, z)$.

$$\therefore \quad \vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$$

is a vector along the normal.

Again,

$$\nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

is normal to the tangent plane at P, so it is parallel to the normal vector $\overrightarrow{PQ}$. Therefore,

$$\vec{R} - \vec{r} = \lambda \nabla F,$$

where λ is a scalar parameter.

$$(X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k} = \lambda \left(\frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k} \right).$$

Hence,

$$\frac{X - x}{\dfrac{\partial F}{\partial x}} = \frac{Y - y}{\dfrac{\partial F}{\partial y}} = \frac{Z - z}{\dfrac{\partial F}{\partial z}} = \lambda$$

Ex.1. Find the equation of the tangent plane and normal to the surface

$$z = x^2 + y^2$$

at the point $(1, -1, 2)$.

Solution. Equation of the surface is

$$F(x, y, z) = z - x^2 - y^2 = 0.$$

$$\frac{\partial F}{\partial x} = -2x = -2 \quad \text{at } (1, -1, 2),$$

$$\frac{\partial F}{\partial y} = -2y = 2 \quad \text{at } (1, -1, 2),$$

$$\frac{\partial F}{\partial z} = 1.$$

Therefore, the equation of the tangent plane at the point $(1, -1, 2)$ is

$$(x-1)(-2)+(y+1)(2)+(z-2)(1)=0.$$

$$-2x+2+2y+2+z-2=0,$$

$$-2x+2y+z+2=0,$$

or

$$2x-2y-z=2.$$

Equation of the Normal is

$$\frac{X-1}{-2}=\frac{Y+1}{2}=\frac{Z-2}{1}.$$

Ex.2. Find the equation of the tangent plane and normal to the surface

$$xyz = 4$$

at the point $(1, 2, 2)$.

Solution. Equation of the surface is

$$F(x, y, z) = xyz - 4 = 0.$$

$$\frac{\partial F}{\partial x} = yz = 4, \quad \frac{\partial F}{\partial y} = xz = 2, \quad \frac{\partial F}{\partial z} = xy = 2 \quad \text{at } (1, 2, 2).$$

Therefore, the equation of the tangent plane at $(1, 2, 2)$ is

$$(x - 1)(4) + (y - 2)(2) + (z - 2)(2) = 0,$$

$$4x - 4 + 2y - 4 + 2z - 4 = 0,$$

$$4x + 2y + 2z = 12,$$

or

$$2x + y + z = 6.$$

Equation of the normal at $(1, 2, 2)$ is

$$\frac{x-1}{4} = \frac{y-2}{2} = \frac{z-2}{2}, \quad \text{or} \quad \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}.$$

Ex.3. Prove that at points common to the surface

$$a(yz + zx + xy) = xyz$$

and a sphere whose centre is the origin, the tangent plane to the surface makes intercepts on the axes whose sum is constant.

Solution. Equation of the given surface is

$$F(x, y, z) = a(yz + zx + xy) - xyz = 0.$$

$$\frac{\partial F}{\partial x} = a(y + z) - yz, \quad \frac{\partial F}{\partial y} = a(x + z) - xz,$$

$$\frac{\partial F}{\partial z} = a(x + y) - xy.$$

Equation of the sphere with centre at origin and radius r is

$$x^2 + y^2 + z^2 = r^2. \tag{2}$$

At a common point $P(x, y, z)$, the equation of the tangent plane is

$$(X - x)[a(y + z) - yz] + (Y - y)[a(x + z) - xz] + (Z - z)[a(x + y) - xy] = 0.$$

or

$$X[a(y + z) - yz] + Y[a(x + z) - xz] + Z[a(x + y) - xy] = 2a(yz + zx + xy) - 3xyz.$$

Using $a(yz + zx + xy) = xyz$, we get

$$X[a(y + z) - yz] + Y[a(x + z) - xz] + Z[a(x + y) - xy] = 2xyz - 3xyz = -xyz.$$

Dividing throughout appropriately, we obtain intercept form

$$\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z} = \text{constant}.$$

Hence, the sum of intercepts is constant.

First Fundamental Form (Metric of the Surface)

Let a surface be represented parametrically by

$$\vec{r} = \vec{r}(u, v)$$

where u and v are parameters. Let,

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u}, \qquad \vec{r}_2 = \frac{\partial \vec{r}}{\partial v}$$

These vectors are tangent to the surface.

Let,

$$E = \vec{r}_1 \cdot \vec{r}_1, \qquad F = \vec{r}_1 \cdot \vec{r}_2, \qquad G = \vec{r}_2 \cdot \vec{r}_2$$

Then the quadratic differential form

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

is called the **first fundamental form** or the **metric** of the surface, where du, dv are differential elements.

The quantities E, F, and G are called the first fundamental coefficients or first order fundamental magnitudes.

The first fundamental form is used to determine:

- Length of curves on the surface
- Angle between curves
- Area of surfaces

Unit Normal to the Surface

The unit normal vector to the surface is

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

Let,

$$H = |\vec{r}_1 \times \vec{r}_2|$$

then

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$$

Second Fundamental Form

Let us consider the second derivatives of $\vec{r}$:

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2}, \qquad \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v} = \vec{r}_{21}, \qquad \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2}$$

Let,

$$L = \vec{r}_{11} \cdot \hat{N} \qquad M = \vec{r}_{12} \cdot \hat{N} \qquad N = \vec{r}_{22} \cdot \hat{N}$$

Then the quadratic differential form

$$L\,du^2 + 2M\,du\,dv + N\,dv^2$$

is called the **second fundamental form** of the surface. The quantities L, M, and N are called the second fundamental coefficients or second order fundamental magnitudes.

It should be noted that L, M, and N represent the components of $\vec{r}_{11}$, $\vec{r}_{12}$ and $\vec{r}_{22}$ respectively in the direction of the unit normal $\hat{N}$ to the surface.

Remark:

- The first fundamental form describes the **intrinsic geometry** of the surface.
- The second fundamental form describes the **curvature properties** of the surface.

THANK YOU

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