Space Curves - Differential Geometry

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MATHEMATICAL EXPLORATIONS

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Intrinsic Equations of a Space Curve:

In differential geometry, the **intrinsic equations** of a curve describe the geometry of the curve independently of its position and orientation in space. These equations depend only on quantities defined along the curve itself.

For a space curve parameterised by arc length s,

$$\vec{r} = \vec{r}(s),$$

the intrinsic equations are given in terms of:

$$\boxed{\kappa = \kappa(s), \qquad \tau = \tau(s)}$$

where,

- $\kappa(s) =$ curvature,
- $\tau(s) =$ torsion.

Fundamental Theorems for Space Curves:

Theorem 1 (Existence Theorem). If $\kappa(s)$ and $\tau(s)$ are continuous functions of a real variable s ($s \geq 0$), then there exists a space curve for which κ is the curvature, τ is the torsion, and s is the arc length measured from some suitable base point.

Proof. From the existence theorem on linear differential equations, the system

$$\frac{d\alpha}{ds} = \kappa\beta,$$
$$\frac{d\beta}{ds} = \tau\gamma - \kappa\alpha,$$
$$\frac{d\gamma}{ds} = -\tau\beta \tag{1}$$

admits a unique set of solutions for a given set of initial values α, β, γ at $s = 0$.

Hence there exists a unique set $\alpha_1, \beta_1, \gamma_1$ with

$$\alpha_1(0) = 1, \quad \beta_1(0) = 0, \quad \gamma_1(0) = 0.$$

Similarly, there exists a unique set $\alpha_2, \beta_2, \gamma_2$ with

$$\alpha_2(0) = 0, \quad \beta_2(0) = 1, \quad \gamma_2(0) = 0,$$

and another unique set $\alpha_3, \beta_3, \gamma_3$ with

$$\alpha_3(0) = 0, \quad \beta_3(0) = 0, \quad \gamma_3(0) = 1.$$

Now,

$$\begin{aligned} \frac{d}{ds}(\alpha_1^2 + \beta_1^2 + \gamma_1^2) &= 2\left(\alpha_1 \frac{d\alpha_1}{ds} + \beta_1 \frac{d\beta_1}{ds} + \gamma_1 \frac{d\gamma_1}{ds}\right) \\ &= 2\left[\alpha_1(\kappa\beta_1) + \beta_1(\tau\gamma_1 - \kappa\alpha_1) + \gamma_1(-\tau\beta_1)\right] = 0 \end{aligned}$$

Integrating, we get

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = C_1 \quad \text{(constant)}$$

Initially at $s = 0, \quad \alpha_1 = 1, \quad \beta_1 = 0, \quad \gamma_1 = 0, \Rightarrow C_1 = 1$
Therefore, $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1.$
Similarly,

$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1, \qquad \alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1. \tag{2}$$

Again,

$$\begin{aligned}
\frac{d}{ds}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2) &= \alpha_1\frac{d\alpha_2}{ds} + \beta_1\frac{d\beta_2}{ds} + \gamma_1\frac{d\gamma_2}{ds} \\
&\quad + \frac{d\alpha_1}{ds}\alpha_2 + \frac{d\beta_1}{ds}\beta_2 + \frac{d\gamma_1}{ds}\gamma_2 \\
&= \alpha_1(\kappa\beta_2) + \beta_1(\tau\gamma_2 - \kappa\alpha_2) + \gamma_1(-\tau\beta_2) \\
&\quad + (\kappa\beta_1)\alpha_2 + (\tau\gamma_1 - \kappa\alpha_1)\beta_2 + (-\tau\beta_1)\gamma_2 \\
&= 0
\end{aligned}$$

Integrating,

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = C_2 \quad \text{(constant)}.$$

Using initial values,

$$\alpha_1(0) = 1,\ \beta_1(0) = 0,\ \gamma_1(0) = 0,$$

$$\alpha_2(0) = 0,\ \beta_2(0) = 1,\ \gamma_2(0) = 0,$$

we obtain $C_2 = 0$. Thus,

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0. \tag{3}$$

Similarly,

$$\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0, \qquad \alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 = 0.$$

Thus we have six equations given by (2) and (3) in the elements of three sets namely $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$. Hence it follows that there are three mutually orthogonal unit vectors

$$\hat{t} = (\alpha_1, \alpha_2, \alpha_3), \qquad \hat{n} = (\beta_1, \beta_2, \beta_3), \qquad \hat{b} = (\gamma_1, \gamma_2, \gamma_3),$$

defined for each value of s.

Now let the curve be defined by

$$\vec{r}(s) = \int^s \hat{t}(s)\, ds. \tag{4}$$

Now,

$$|\vec{r}\,'| = |\hat{t}| = 1, \qquad \text{and} \qquad \hat{t}' = \kappa \hat{n}, \quad \text{where } |\hat{n}| = 1.$$

Thus κ is the curvature of the curve given by (4).

Again,

$$\hat{b} = \hat{t} \times \hat{n}.$$

Differentiating,

$$\begin{aligned}\hat{b}' &= \hat{t}' \times \hat{n} + \hat{t} \times \hat{n}' \\ &= \kappa(\hat{n} \times \hat{n}) + \hat{t} \times (-\kappa\hat{t} + \tau\hat{n}) \qquad \text{(putting for } \hat{t}' \text{ and } \hat{n}'\text{)} \\ &= \kappa(\hat{n} \times \hat{n}) - \kappa(\hat{t} \times \hat{t}) + \tau(\hat{t} \times \hat{n}) \\ &= -\tau\hat{n},\end{aligned}$$

since $\hat{n} \times \hat{n} = 0$ and $\hat{t} \times \hat{t} = 0$, and $\hat{t} \times \hat{n} = \hat{b}$.
Hence $|\hat{b}| = 1$ and τ is the torsion of the curve.

Thus there exists a curve given by (4) where $\hat{t}, \hat{n}, \hat{b}$ are unit vectors along tangent, principal normal and binormal respectively, and κ and τ are its curvature and torsion.

Fundamental Theorems for Space Curves:

Theorem 2 (Uniqueness Theorem). A curve is uniquely determined, except as to position in space, when its curvature and torsion are given functions of its arc length s.

Proof. If possible, let C and C_1 be two curves having equal curvature and torsion for the same values of s. We shall use the suffix unity for quantities belonging to C_1.

Let C_1 be moved (without deformation) so that the two points on C and C_1 corresponding to the same value of s coincide. Let C_1 be rotated in such a manner that at $s = 0$ the two triads $(\hat{t}, \hat{n}, \hat{b})$ and $(\hat{t}_1, \hat{n}_1, \hat{b}_1)$ coincide.

Now differentiating the products $\hat{t} \cdot \hat{t}_1$ etc., and using the Frenet formulae, we have

$$\frac{d}{ds}(\hat{t} \cdot \hat{t}_1) = \hat{t} \cdot \hat{t}_1' + \hat{t}' \cdot \hat{t}_1 = \kappa \hat{t} \cdot \hat{n}_1 + \kappa \hat{n} \cdot \hat{t}_1, \tag{1}$$

$$\begin{aligned}\frac{d}{ds}(\hat{n}\cdot\hat{n}_1) &= \hat{n}\cdot\hat{n}_1' + \hat{n}'\cdot\hat{n}_1 \\ &= \hat{n}\cdot(\tau\hat{b}_1 - \kappa\hat{t}_1) + (-\kappa\hat{t} + \tau\hat{b})\cdot\hat{n}_1, \qquad (2)\end{aligned}$$

$$\begin{aligned}\frac{d}{ds}(\hat{b}\cdot\hat{b}_1) &= \hat{b}\cdot\hat{b}_1' + \hat{b}'\cdot\hat{b}_1 \\ &= \hat{b}\cdot(-\tau\hat{n}_1) + (-\tau\hat{n})\cdot\hat{b}_1. \qquad (3)\end{aligned}$$

Adding (1), (2) and (3), we get

$$\frac{d}{ds}\left(\hat{t}\cdot\hat{t}_1 + \hat{n}\cdot\hat{n}_1 + \hat{b}\cdot\hat{b}_1\right) = 0.$$

On integrating,

$$\hat{t} \cdot \hat{t}_1 + \hat{n} \cdot \hat{n}_1 + \hat{b} \cdot \hat{b}_1 = \text{constant}$$

At $s = 0,\ t = t_1,\ n = n_1,\ b = b_1$, which gives

$$t \cdot t_1 = n \cdot n_1 = b \cdot b_1 = 1.$$

Thus the value of the constant is 3.
Therefore

$$t \cdot t_1 + n \cdot n_1 + b \cdot b_1 = 3$$

or

$$\cos\alpha + \cos\beta + \cos\gamma = 3 \tag{4}$$

where α, β, γ are the angles between t, t_1; n, n_1; and b, b_1.
Equation (4) gives the sum of three cosines equal to 3, which is only possible when each angle is zero or an integral multiple of 2π.

Thus for each pair of corresponding points

$$t = t_1, \quad n = n_1, \quad b = b_1$$

Also $t = t_1$ gives $r' = r_1'$.
i.e.,

$$\frac{d}{ds}(r - r_1) = 0 \quad \text{or} \quad r - r_1 = a \quad \text{(constant vector)}$$

But at $s = 0$, $r = r_1$, so the constant vector $a = 0$.
Hence $r = r_1$ at all corresponding points, so the two curves C and C_1 are identical, i.e., they coincide having the same curvatures and torsions at the corresponding points. We can say that a curve is uniquely determined when curvature and torsion are given.

Ex.1. Show that the intrinsic equations of the curve given by

$$x = ae^u \cos u, \quad y = ae^u \sin u, \quad z = be^u$$

are

$$\kappa = \frac{a\sqrt{2}}{s\sqrt{(2a^2+b^2)}}, \qquad \tau = \frac{b}{s\sqrt{(2a^2+b^2)}}.$$

Solution. Here

$$\mathbf{r} = (ae^u \cos u,\ ae^u \sin u,\ be^u)$$

$$\therefore \dot{\mathbf{r}} = [ae^u(\cos u - \sin u),\ ae^u(\sin u + \cos u),\ be^u]$$

$$|\dot{\mathbf{r}}| = \dot{s} = e^u\sqrt{a^2(\cos u - \sin u)^2 + a^2(\sin u + \cos u)^2 + b^2}$$

$$= e^u\sqrt{2a^2+b^2}$$

$$s = \int_{-\infty}^{u} |\dot{\mathbf{r}}|\, du = \int_{-\infty}^{u} e^u\sqrt{(2a^2+b^2)}\, du = e^u\sqrt{(2a^2+b^2)} = \dot{s} \tag{1}$$

$$\mathbf{r}' = \frac{\dot{\mathbf{r}}}{\dot{s}} = \frac{[a(\cos u - \sin u),\ a(\sin u + \cos u),\ b]}{\sqrt{(2a^2 + b^2)}}$$

$$\mathbf{r}'' = \kappa \mathbf{n} = \frac{[-a(\sin u + \cos u),\ a(\cos u - \sin u),\ 0]}{\sqrt{(2a^2 + b^2)}} \cdot \frac{1}{s}, \tag{2}$$

Taking modulus of both sides, we get

$$\kappa = |\mathbf{r}''| = \frac{a\sqrt{2}}{\sqrt{(2a^2 + b^2)}\, s}, \qquad \left[\because \frac{1}{\dot{s}} = \frac{1}{s} \text{ from } (1)\right]$$

Also from (2)

$$s\mathbf{r}'' = \frac{[-a(\sin u + \cos u),\ a(\cos u - \sin u),\ 0]}{\sqrt{(2a^2 + b^2)}}$$

Differentiating w.r.t. s, we get

$$s\mathbf{r}''' + \mathbf{r}'' = \frac{[-a(\cos u - \sin u),\ -a(\sin u + \cos u),\ 0]}{\sqrt{(2a^2 + b^2)}} \cdot \frac{1}{s}, \qquad [\because \text{ from } (1)]$$

or,

$$s^2\mathbf{r}''' + s\mathbf{r}'' = \frac{[-a(\cos u - \sin u),\ -a(\sin u + \cos u),\ 0]}{\sqrt{(2a^2 + b^2)}}$$

Now

$$[\mathbf{r}', s\mathbf{r}'', s^2\mathbf{r}''' + s\mathbf{r}''] = \frac{1}{(2a^2+b^2)^{3/2}} \begin{vmatrix} a(\cos u - \sin u) & a(\sin u + \cos u) & b \\ -a(\sin u + \cos u) & a(\cos u - \sin u) & 0 \\ -a(\cos u - \sin u) & -a(\sin u + \cos u) & 0 \end{vmatrix}$$

$$\Rightarrow s^3\kappa^2\tau = \frac{1}{(2a^2+b^2)^{3/2}} a^2 b \left[(\sin u + \cos u)^2 + (\cos u - \sin u)^2\right]$$

$$\Rightarrow s^3\kappa^2\tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}$$

$$\text{or} \quad s^3 \frac{2a^2}{(2a^2+b^2)} \frac{1}{s^2} \tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}} \qquad [\text{on putting the value of } \kappa]$$

$$\Rightarrow \tau = \frac{b}{\sqrt{(2a^2 + b^2)}} \cdot \frac{1}{s}$$

Hence, the intrinsic equations of the given curve are

$$\kappa = \frac{\sqrt{2}\,a}{\sqrt{(2a^2 + b^2)}} \cdot \frac{1}{s}, \qquad \tau = \frac{b}{\sqrt{(2a^2 + b^2)}} \cdot \frac{1}{s}.$$

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