# Space Curves - Differential Geometry

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MATHEMATICAL EXPLORATIONS

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Ex.1. Find the curvature and torsion for the curve

$$x = a\cos t, \quad y = a\sin t, \quad z = at\cot\alpha.$$

**Solution.** Position vector $\vec{r}$ in terms of parameter $t$ is given by

$$\vec{r} = (a\cos t,\ a\sin t,\ at\cot\alpha).$$

$$\dot{\vec{r}} = (-a\sin t,\ a\cos t,\ a\cot\alpha),$$

$$\ddot{\vec{r}} = (-a\cos t,\ -a\sin t,\ 0),$$

$$\dddot{\vec{r}} = (a\sin t,\ -a\cos t,\ 0).$$

$$\therefore \quad \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\cot\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \left(a^2\sin t\cot\alpha,\ -a^2\cos t\cot\alpha,\ a^2\right).$$

$$\left|\dot{\vec{r}} \times \ddot{\vec{r}}\right| = a^2\csc\alpha.$$

Also,

$$|\dot{\vec{r}}| = \left(a^2\sin^2 t + a^2\cos^2 t + a^2\cot^2\alpha\right)^{1/2} = a\csc\alpha.$$

Again,

$$[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = \begin{vmatrix} -a\sin t & a\cos t & a\cot\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = a^3\cot\alpha.$$

$$\therefore \quad \kappa = \frac{\left|\dot{\vec{r}} \times \ddot{\vec{r}}\right|}{|\dot{\vec{r}}|^3} = \frac{a^2\csc\alpha}{a^3\csc^3\alpha} = \frac{1}{a}\sin^2\alpha.$$

and

$$\tau = \frac{[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}]}{\left|\dot{\vec{r}} \times \ddot{\vec{r}}\right|^2} = \frac{a^3\cot\alpha}{a^4\csc^2\alpha} = \frac{1}{a}\sin\alpha\cos\alpha.$$

Ex.2. For the curve

$$x = 4a\cos^3 t, \quad y = 4a\sin^3 t, \quad z = 3c\cos 2t,$$

prove that

$$\kappa = \frac{a}{6(a^2+c^2)\sin 2t}.$$

**Solution.** We have

$$\vec{r} = (4a\cos^3 t,\ 4a\sin^3 t,\ 3c\cos 2t).$$

$$\begin{aligned}\dot{\vec{r}} &= (-12a\cos^2 t\sin t,\ 12a\sin^2 t\cos t,\ -6c\sin 2t)\\ &= (-6a\cos t\sin 2t,\ 6a\sin t\sin 2t,\ -6c\sin 2t).\end{aligned}$$

Hence

$$\dot{\vec{r}} = 6\sin 2t\,(-a\cos t,\ a\sin t,\ -c).$$

Also,

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{r}'\,\dot{s}.$$

$$\Rightarrow \dot{\vec{r}} = \vec{r}'\,\dot{s} \quad \Rightarrow \quad \vec{r}' = \frac{\dot{\vec{r}}}{\dot{s}}.$$

$$|\dot{\vec{r}}| = |\vec{t}|\,\dot{s} = \dot{s} = \frac{ds}{dt}.$$

Therefore

$$\dot{s} = 6\sin 2t\sqrt{a^2 + c^2}.$$

Now

$$\vec{r}' = \frac{\dot{\vec{r}}}{\dot{s}} = \frac{(-a\cos t,\ a\sin t,\ -c)}{\sqrt{a^2 + c^2}}.$$

$$\vec{r}'' = \frac{(a\sin t,\ a\cos t,\ 0)}{\sqrt{a^2 + c^2}}.$$

Hence

$$\kappa\vec{n} = \vec{r}'' = \frac{(a\sin t,\ a\cos t,\ 0)}{\sqrt{a^2+c^2}}.$$

$$\kappa|\vec{n}| = \frac{1}{6\sin 2t(a^2+c^2)}\left(a^2\sin^2 t + a^2\cos^2 t + 0\right)^{1/2}.$$

$$\Rightarrow \kappa = \frac{a}{6\sin 2t(a^2+c^2)}.$$

Ex.3. Find the osculating plane, curvature and torsion at any point of the curve

$$x = a\cos 2u, \quad y = a\sin 2u, \quad z = 2a\sin u.$$

**Solution.** The position vector at any point on the curve is given by

$$\vec{r} = a(\cos 2u,\ \sin 2u,\ 2\sin u).$$

$$\dot{\vec{r}} = 2a(-\sin 2u,\ \cos 2u,\ \cos u),$$

$$\ddot{\vec{r}} = -2a(2\cos 2u,\ 2\sin 2u,\ \sin u),$$

$$\dddot{\vec{r}} = -2a(-4\sin 2u,\ 4\cos 2u,\ \cos u).$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = 4a^2\left[(\sin u + \sin 2u\cos u),\ -(\cos u + \cos u\cos 2u),\ 2\right].[\dot{\vec{r}} \times \ddot{\vec{r}}] \cdot \dddot{\vec{r}}$$

$$= 8a^3\Big[4\sin 2u(\sin u + \sin 2u\cos u) + 4\cos 2u(\cos u + \cos u\cos 2u) - 2\cos u\Big].$$

$$= 8a^3\left[4\cos 2u\cos u + \sin 2u\sin u + 4\cos u(\cos^2 2u + \sin^2 2u) - 2\cos u\right].$$

$$= 8a^3[4\cos u + 4\cos u - 2\cos u] = 48a^3\cos u$$

Again,

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = 4a^2\left[(\sin u + \sin 2u\cos u)^2 + (\cos u + \cos u\cos 2u)^2 + 4\right]^{1/2}.$$

$$= 4a^2\left[\sin^2 u + \cos^2 u + \cos^2 u(\cos^2 2u + \sin^2 2u) + 2\cos u(\cos 2u\cos u + \sin 2u\sin u) + 4\right]^{1/2}.$$

$$= 4a^2\left[5 + 3\cos^2 u\right]^{1/2}$$

The equation of the osculating plane is given by

$$[\vec{R} - \vec{r},\ \dot{\vec{r}},\ \ddot{\vec{r}}] = 0, \quad \text{i.e.,} \quad (\vec{R} - \vec{r}) \cdot (\dot{\vec{r}} \times \ddot{\vec{r}}) = 0$$

Hence,

$$(x - a\cos 2u)(\sin u + \sin 2u\cos u) - (y - a\sin 2u)\cos u(1 + \cos 2u) + 2(z - 2a\sin u) = 0$$

or

$$(\sin u + \sin 2u\cos u)x - 2\cos^3 u\, y + 2z = 3a\sin u$$

$$\therefore \quad \tau = \frac{[\dot{\vec{r}},\ \ddot{\vec{r}},\ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} = \frac{48a^3\cos u}{16a^4(5 + 3\cos^2 u)} = \frac{3}{a(5\sec u + 3\cos u)}$$

Ex.4. Show that the Serret–Frenet formulae can be written in the form

$$\frac{d\hat{t}}{ds} = \vec{\omega} \times \hat{t}, \qquad \frac{d\hat{n}}{ds} = \vec{\omega} \times \hat{n}, \qquad \frac{d\hat{b}}{ds} = \vec{\omega} \times \hat{b}$$

and determine $\vec{\omega}$.

**Solution.** Let $\{\hat{t}, \hat{n}, \hat{b}\}$ denote the orthonormal Serret–Frenet triad of unit vectors,

$$\begin{aligned} \frac{d\hat{t}}{ds} &= \kappa \hat{n} && \text{(Serret–Frenet formula)} \\ &= \kappa(\hat{b} \times \hat{t}) && \text{since } \hat{b} \times \hat{t} = \hat{n} \\ &= \tau(\hat{t} \times \hat{t}) + \kappa(\hat{b} \times \hat{t}) && \text{inserting zero term} \\ &= (\tau \hat{t} + \kappa \hat{b}) \times \hat{t} && \text{since } \hat{t} \times \hat{t} = 0 \end{aligned} \tag{1}$$

$$
\begin{aligned}
\frac{d\hat{n}}{ds} &= -\kappa\hat{t} + \tau\hat{b} \qquad \text{(Serret–Frenet formula)} \\
&= \tau(\hat{t}\times\hat{n}) + \kappa(\hat{b}\times\hat{n}) \qquad \text{using vector identities} \\
&= (\tau\hat{t} + \kappa\hat{b}) \times \hat{n} \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\hat{b}}{ds} &= -\tau\hat{n} \qquad \text{(Serret–Frenet formula)} \\
&= \tau(\hat{t}\times\hat{b}) + \kappa(\hat{b}\times\hat{b}) \qquad \text{since } -\hat{n} = \hat{t}\times\hat{b} \\
&= (\tau\hat{t} + \kappa\hat{b}) \times \hat{b} \qquad \text{since } \hat{b}\times\hat{b} = 0 \qquad (3)
\end{aligned}
$$


$$
\begin{aligned}
\frac{d\hat{b}}{ds} &= -\tau\hat{n} \qquad \text{(Serret–Frenet formula)} \\
&= \tau(\hat{t}\times\hat{b}) + \kappa(\hat{b}\times\hat{b}) \qquad \text{since } -\hat{n} = \hat{t}\times\hat{b} \\
&= (\tau\hat{t} + \kappa\hat{b}) \times \hat{b} \qquad \text{since } \hat{b}\times\hat{b} = 0 \qquad (3)
\end{aligned}
$$

From (1), (2), and (3), we obtain

$$\frac{d\hat{t}}{ds} = \vec{\omega} \times \hat{t}, \qquad \frac{d\hat{n}}{ds} = \vec{\omega} \times \hat{n}, \qquad \frac{d\hat{b}}{ds} = \vec{\omega} \times \hat{b},$$

where the angular velocity vector of the moving triad is

$$\vec{\omega} = \tau\,\hat{t} + \kappa\,\hat{b}$$

Ex.5. Show that

$$\mathbf{r}''' = \kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b}$$

and hence show that

$$\mathbf{r}'''' = (\kappa'' - \kappa^3 - \kappa\tau^2)\hat{n} - 3\kappa\kappa'\hat{t} + (2\kappa'\tau + \kappa\tau')\hat{b}.$$

**Solution.** Let $\vec{r}(s)$ be a space curve parameterised by arc length $s$. Let $\{\hat{t}, \hat{n}, \hat{b}\}$ denote the Serret–Frenet orthonormal triad.

$$\vec{r}'' = \kappa\hat{n} \qquad \text{(since } \vec{r}' = \hat{t},\ \hat{t}' = \kappa\hat{n}\text{)} \tag{1}$$

$$\begin{aligned}
\vec{r}''' &= \frac{d}{ds}(\kappa\hat{n}) \qquad \text{(product rule)} \\
&= \kappa'\hat{n} + \kappa\hat{n}' \qquad \text{(since } \kappa = \kappa(s)\text{)} \\
&= \kappa'\hat{n} + \kappa(\tau\hat{b} - \kappa\hat{t}) \qquad \text{(Serret–Frenet: } \hat{n}' = -\kappa\hat{t} + \tau\hat{b}\text{)} \\
&= \kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b}. \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\vec{r}'''' &= \frac{d}{ds}\left(\kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b}\right) \qquad \text{(differentiate term by term)} \\
&= \kappa''\hat{n} + \kappa'\hat{n}' - 2\kappa\kappa'\hat{t} - \kappa^2\hat{t}' + \kappa'\tau\hat{b} + \kappa\tau'\hat{b} + \kappa\tau\hat{b}' \qquad \text{(product rule applied).} \qquad (3)
\end{aligned}$$

Using

$$\hat{t}' = \kappa\hat{n}, \qquad \hat{n}' = -\kappa\hat{t} + \tau\hat{b}, \qquad \hat{b}' = -\tau\hat{n},$$

we obtain

$$\begin{aligned}\vec{r}'''' &= \kappa''\hat{n} + \kappa'(\tau\hat{b} - \kappa\hat{t}) - 2\kappa\kappa'\hat{t} - \kappa^3\hat{n} + \kappa'\tau\hat{b} + \kappa\tau'\hat{b} - \kappa\tau^2\hat{n} \\ &= \left(\kappa'' - \kappa^3 - \kappa\tau^2\right)\hat{n} - 3\kappa\kappa'\hat{t} + \left(2\kappa'\tau + \kappa\tau'\right)\hat{b}\end{aligned}$$

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