Space Curves - Differential Geometry

Presented by

MATHEMATICAL EXPLORATIONS

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Normal Lines:

The normal line at point P to the given curve is a line perpendicular to the tangent at point P to the curve. Obviously for a three–dimensional space curve there will be an infinite number of such normal lines.

Let a space curve be given by

$$\vec{r} = \vec{r}(t),$$

and let P be the point on the curve corresponding to the parameter value $t = t_0$.

The tangent vector to the curve at the point P is

$$\vec{r}\,'(t_0) = \left.\frac{d\vec{r}}{dt}\right|_{t=t_0}.$$

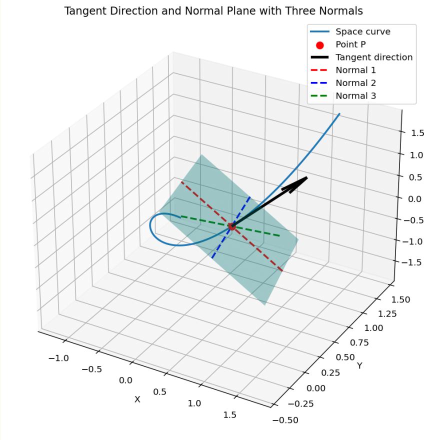
This vector determines the direction of the tangent line at P.

A *normal line* at the point P is defined as a line passing through P whose direction is perpendicular to the tangent at P. If $\vec{a}$ is the direction vector of such a line, then

$$\vec{a} \cdot \vec{r}'(t_0) = 0.$$

In two dimensions, there is only one direction perpendicular to a given tangent, so the normal line is unique.
In three–dimensional space, all vectors perpendicular to the tangent vector $\vec{r}'(t_0)$ lie in a plane perpendicular to the tangent at P. Since this plane contains infinitely many directions, there exist infinitely many vectors $\vec{a}$ satisfying the condition

$$\vec{a} \cdot \vec{r}'(t_0) = 0.$$



Hence, for a three–dimensional space curve, there are infinitely many normal lines at a given point P.

Normal plane:

The normal plane at point P to the given curve is the plane passing through the point P and perpendicular to the tangent at P.
Thus we can say that the normal plane at point P on the space curve and let $\vec{R}$ be the position vector of any current point on the normal plane at P, then the vector $(\vec{R}-\vec{r})$ lies in the plane. Since the vector $\vec{r}$ is perpendicular to this plane, we have

$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0 \tag{1}$$

which is the equation of the normal plane at point P.
Again the equation (1) can be put in the form

$$(\vec{R}-\vec{r})\cdot\hat{t}=0 \tag{2}$$

Cartesian Equivalent

Let

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}, \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

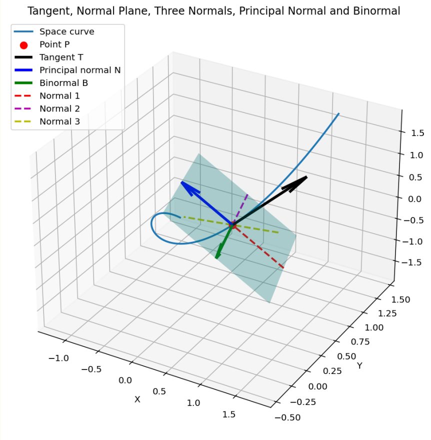
Putting these in (1), we get

$$[(X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}] \cdot [\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}] = 0$$

or,

$$(X - x)\dot{x} + (Y - y)\dot{y} + (Z - z)\dot{z} = 0$$

Or, all the normals, two normals are of special significance. These are known as *Principal normal* and *Binormal*, which are defined below.


Tangent, Normal Plane, Three Normals, Principal Normal and Binormal
Space curve
Point P
Tangent T
Principal normal N
Binormal B
Normal 1
Normal 2
Normal 3
X
Y
−1.0
−0.5
0.0
0.5
1.0
1.5
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5

Principal Normal

The normal lying in the osculating plane at a point P on the space curve is called the principal normal at point P.

Thus we can say that the principal normal at any point P to a given curve is the line of intersection of the normal plane at P and the osculating plane at P. We shall denote the unit vector along the principal normal by $\hat{n}$.

Binormal

The normal perpendicular to the principal normal at point P is called binormal at point P.

Thus we can say that the binormal at any point P is the line perpendicular to the osculating plane at P. The unit vector along the binormal is denoted by $\hat{b}$ and we choose the sense of $\hat{b}$ in such a manner that the triad $\hat{t}, \hat{n}, \hat{b}$ form a right–handed system, i.e.,

$$\hat{b} = \hat{t} \times \hat{n}.$$

Directions of Principal Normal and Binormal

Since the binormal is perpendicular to the osculating plane, therefore it must be parallel to the vector $\dot{\vec{r}} \times \ddot{\vec{r}}$.

In case the general parameter t is replaced by the arc parameter s, then $\dot{\vec{r}} = \vec{r}'$, $\ddot{\vec{r}} = \vec{r}''$, thus the binormal is parallel to the vector $\vec{r}' \times \vec{r}''$.

Again the principal normal is perpendicular to both the tangent and binormal and therefore it must be parallel to the vector

$$(\dot{\vec{r}} \times \ddot{\vec{r}}) \times \dot{\vec{r}},$$

i.e., parallel to the vector

$$(\dot{\vec{r}}^2)\ddot{\vec{r}} - (\dot{\vec{r}} \cdot \ddot{\vec{r}})\dot{\vec{r}}.$$

In case the parameter is s, then

$$\dot{\vec{r}}^2 = \vec{r}'^2 = 1,$$

thus on differentiating,

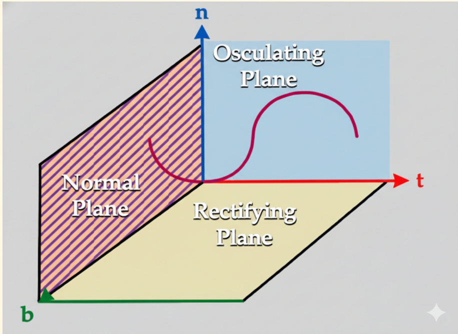
$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = \vec{r}\,' \cdot \vec{r}\,'' = 0.$$

Consequently the principal normal is parallel to $\vec{r}\,''$.

Rectifying Plane

The plane containing the tangent and binormal at P is called the *rectifying plane* at P, i.e., it is the plane passing through P and perpendicular to the principal normal at P. Evidently the equation of this plane is

$$(\vec{R} - \vec{r}) \cdot \hat{n} = 0$$



Orthonormal Triad of Fundamental Unit Vectors $\hat{t}, \hat{n}, \hat{b}$

We have defined a set of three mutually perpendicular unit vectors $\hat{t}, \hat{n}, \hat{b}$ associated with each point of a curve. This set of unit orthonormal triad forms a moving trihedral at point P (say) such that

$$\hat{t} \cdot \hat{n} = 0, \qquad \hat{n} \cdot \hat{b} = 0, \qquad \hat{b} \cdot \hat{t} = 0,$$

$$\hat{n} \times \hat{b} = \hat{t}, \qquad \hat{b} \times \hat{t} = \hat{n}, \qquad \hat{t} \times \hat{n} = \hat{b}.$$

The vectors $\hat{t}, \hat{n}, \hat{b}$ are called *fundamental unit vectors.*

Fundamental Planes

The three planes, osculating plane, normal plane and rectifying plane associated with each point of a curve are called *fundamental planes.* These planes are mutually perpendicular and are determined by the moving trihedral $\hat{t}, \hat{n}, \hat{b}$ at the point.
The equations of fundamental planes are:

Osculating plane:

It contains $\hat{t}$ and $\hat{n}$ and is normal to $\hat{b}$, its equation is

$$(\vec{R} - \vec{r}) \cdot \hat{b} = 0.$$

Normal plane:

It contains $\hat{n}$ and $\hat{b}$ and is normal to $\hat{t}$, its equation is

$$(\vec{R} - \vec{r}) \cdot \hat{t} = 0.$$

Rectifying plane:

It contains $\hat{b}$ and $\hat{t}$ and is normal to $\hat{n}$, its equation is

$$(\vec{R} - \vec{r}) \cdot \hat{n} = 0.$$

Equation of the Principal Normal and Binormal

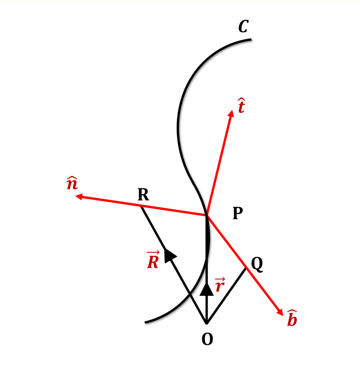
Let $\vec{r}$ be the position vector of any point P on the given curve C. Let $\vec{R}$ be the position vector of a current point R on the principal normal, then we have $\overrightarrow{OP} = \vec{r}, \overrightarrow{OR} = \vec{R}$ and $\overrightarrow{PR} = \lambda\hat{n}$, since $\hat{n}$ is the unit vector along the principal normal and λ is some scalar.

By triangle law of vectors, we have

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} \quad \Rightarrow \quad \vec{R} = \vec{r} + \lambda\hat{n},$$

which is the required equation of the principal normal.

Similarly if $\vec{R}$ is the position vector of a current point Q on the binormal, then the equation of binormal is given by



$$\vec{R} = \vec{r} + \mu\hat{b}, \quad \text{where } \mu \text{ is a scalar.}$$

Curvature:

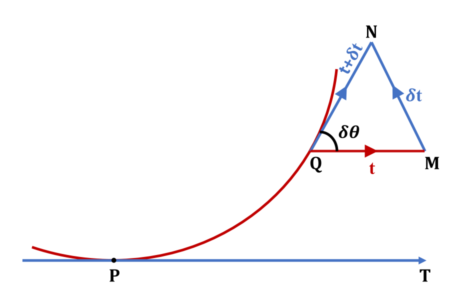
The curvature at a point P of a given curve is the arc rate of rotation of the tangent (i.e., change in the direction of tangent) at P. Its magnitude is denoted by κ (kappa).

To find an expression for the curvature (κ) at a given point P of a given curve:

Let Q be a point very near to P on the curve. Arc PQ is δs and let the direction of tangent at Q makes an angle $\delta\theta$ with the direction of tangent at P. Again the unit tangent vector is not constant vector, since its direction changes from point to point.

Let t and $t+\delta t$ be its values at P and Q respectively.

If $\overrightarrow{QM} = t$ and $\overrightarrow{QN} = t + \delta t$, then we have



$$\overrightarrow{MN} = \delta t, \qquad \angle MQN = \delta\theta \quad \text{and} \quad |\overrightarrow{QM}| = |\overrightarrow{QN}| = 1$$

From the isosceles triangle QMN, we have

$$MN = 2QM \sin\left(\frac{1}{2}\delta\theta\right) = 2\sin\left(\frac{1}{2}\delta\theta\right)$$

$$\Rightarrow \quad |\delta\vec{t}| = 2\sin\left(\frac{1}{2}\delta\theta\right)$$

$$\Rightarrow \quad \left|\frac{\delta t}{\delta\theta}\right| = \frac{\sin\left(\frac{1}{2}\delta\theta\right)}{\frac{1}{2}\delta\theta}$$

Taking limits,

$$\left|\frac{dt}{d\theta}\right| = 1 \tag{1}$$

$\therefore$ **Curvature at P,**

$$\kappa = \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds} \quad \text{along the direction of the tangent.}$$

$$\kappa = \frac{d\theta}{|dt|} \frac{|dt|}{ds} = \left|\frac{d\theta}{dt}\right| \left|\frac{dt}{ds}\right| = \left|\frac{dt}{ds}\right| = \left|\frac{d\vec{r}'}{ds}\right| = |\vec{r}''| \qquad \text{[Using (1)]}$$

which implies that the curvature is the scalar measure of the arc rate of turning of the unit vector t. The reciprocal of κ, i.e., $\frac{1}{\kappa}$, is called the *radius of curvature* and is denoted by ρ.

Deduction:

$$|\vec{r}'| = 1 \quad \vec{r}'^2 = 1$$

Differentiating, we get

$$2\vec{r}' \cdot \vec{r}'' = 0,$$

i.e., $\vec{r}''$ is perpendicular to $\vec{r}'$, i.e., to t.
But $\vec{r}''$ at P lies in the osculating plane at P, or $\vec{r}''$ is a vector in the osculating plane perpendicular to t, implying that $\vec{r}''$ is collinear with $\hat{n}$.
Also $|\vec{r}''| = \kappa$, so we have $\vec{r}'' = \pm\kappa\hat{n}$.
We choose the direction of $\hat{n}$ such that curvature κ is always positive, i.e., we take

$$\vec{r}'' = \kappa\hat{n} \quad \text{or} \quad \frac{dt}{ds} = \kappa\hat{n}$$

Theorem. A necessary and sufficient condition for the curve to be a straight line is that the curvature $\kappa = 0$ at all points of the curve.

Proof. The equation of a straight line in vector form is given by

$$\vec{r} = s\vec{a} + \vec{b},$$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Hence,

$$\vec{t} = \vec{r}' = \vec{a} \quad \text{and} \quad \vec{t}' = \vec{r}'' = 0$$

$$\kappa = \left|\mathbf{r}''\right| = 0$$

i.e., if the curve is a straight line, then $\kappa = 0$, i.e., $\kappa = 0$ is a necessary condition for a curve to be a straight line.

Converse. In case $\kappa = 0$ for all points on the curve, then

$$\vec{r}'' = 0 \tag{1}$$

Integrating we get, $\vec{r}' = \vec{a}$.
Again on integration,

$$\vec{r} = \vec{a}s + \vec{b}. \tag{2}$$

where $\vec{a}$ and $\vec{b}$ are arbitrary constant vectors. The equation (2) represents a straight line for all values of $\vec{a}$ and $\vec{b}$. Hence the curve is a straight line.

Torsion:

Torsion at point P of a given curve is the arc rate of the change in the direction of the binormal at P. Its magnitude is denoted by τ.

To find an expression for the torsion at a point P of a given curve:

Let Q be a point contiguous to P on the curve. Arc $PQ = \delta s$, $\hat{b}$ and $\hat{b} + \delta\hat{b}$ are the unit binormal vectors at P and Q respectively and $\delta\theta$ is the angle between $\hat{b}$ and $\hat{b} + \delta\hat{b}$.

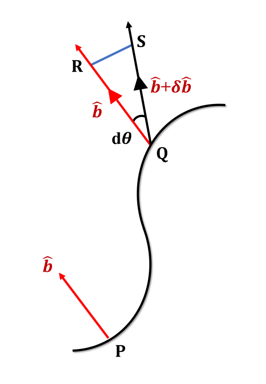
Let,

$$\overrightarrow{QR} = \hat{b}, \qquad \overrightarrow{QS} = \hat{b} + \delta\hat{b},$$

then

$$\overrightarrow{RS} = \delta\hat{b}.$$

Now from the isosceles triangle QRS, we have



$$
\begin{aligned}
RS &= 2QR\sin\frac{1}{2}\delta\theta \\
\Rightarrow |\overrightarrow{RS}| &= 2|\overrightarrow{QR}|\sin\frac{1}{2}\delta\theta \\
\Rightarrow |\delta\hat{b}| &= 2\sin\frac{1}{2}\delta\theta \qquad \because |\overrightarrow{QR}| = 1 \\
\Rightarrow \frac{|\delta\hat{b}|}{|\delta\theta|} &= 2\frac{\sin\frac{1}{2}\delta\theta}{\delta\theta} \\
\Rightarrow \frac{|\delta\hat{b}|}{|\delta\theta|} &= \frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}
\end{aligned}
$$

Taking limits as $\delta\theta \to 0$,

$$\lim_{\delta\theta\to 0} \frac{|\delta\hat{b}|}{|\delta\theta|} = \lim_{\delta\theta\to 0} \frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}$$

$$\Rightarrow \left|\frac{d\hat{b}}{d\theta}\right| = 1$$

Thus, by definition, torsion at P is

$$\tau = \lim_{\delta s\to 0} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$$

$$\tau = \frac{d\theta}{|d\hat{b}|}\frac{|d\hat{b}|}{ds} = \left|\frac{d\theta}{d\hat{b}}\right|\left|\frac{d\hat{b}}{ds}\right| = \left|\frac{d\hat{b}}{ds}\right| = |\hat{b}'|$$

Hence,

$$\tau = |\hat{b}'|.$$

Thus τ is the scalar measure of the arc rate of the unit vector $\vec{b}$. The reciprocal of the torsion is called the *radius of torsion* and is denoted by σ and $\sigma = \frac{1}{\tau}$.

Deduction: We have $\hat{t} \cdot \hat{b} = 0$.
On differentiating,

$$\begin{aligned} &\hat{t} \cdot \hat{b}' + \hat{t}' \cdot \hat{b} = 0 \\ &\Rightarrow \hat{t} \cdot \hat{b}' + \kappa\hat{n} \cdot \hat{b} = 0 \quad [\because \hat{t}' = \kappa\hat{n}] \\ &\Rightarrow \hat{t} \cdot \hat{b}' = 0 \quad [\because \hat{n} \cdot \hat{b} = 0] \\ &\Rightarrow \hat{b} \text{ is perpendicular to } \hat{t} \end{aligned}$$

Further,

$$\hat{b} \cdot \hat{b} = 1 \Rightarrow 2\,\hat{b} \cdot \hat{b}' = 0 \Rightarrow \hat{b}' \text{ is perpendicular to } \hat{b}$$

Thus $\hat{b}'$ is normal to the plane containing $\hat{t}$ and $\hat{b}$, i.e., to the rectifying plane. Hence $\hat{b}'$ is collinear with $\hat{n}$.
Thus

$$\hat{b}' = \pm\tau\hat{n}$$

Since the triad $(\hat{t}, \hat{n}, \hat{b})$ forms a right–handed orthonormal system, the negative sign is taken, i.e.,

$$\hat{b}' = -\tau\hat{n} \quad \text{or} \quad \frac{d\hat{b}}{ds} = -\tau\hat{n}.$$

Theorem: A necessary and sufficient condition that a given curve is a plane curve is that $\tau = 0$ at all points.

Proof. Let the curve be a plane curve. Then the tangent and normal at all points of the curve lie in the plane of the curve, i.e., the plane of the curve is the osculating plane at all points. This implies that the unit vector $\hat{b}$ along the binormal is constant. Hence

$$\frac{d\hat{b}}{ds} = 0 \quad \text{or} \quad \tau = 0.$$

Hence the condition is necessary.

Converse. Let $\tau = 0$ at all points of the curve. This implies that

$$\frac{d\hat{b}}{ds} = 0,$$

i.e., $\hat{b}$ is a constant vector.
Again,

$$\frac{d}{ds}(\vec{r}\cdot\hat{b}) = \frac{d\vec{r}}{ds}\cdot\hat{b} + \vec{r}\cdot\frac{d\hat{b}}{ds} = \hat{t}\cdot\hat{b} + \vec{r}\cdot\hat{b}'.$$

As $\hat{t}$ and $\hat{b}$ are orthogonal, we have $\hat{t}\cdot\hat{b} = 0$. Also $\hat{b}' = 0$.

Therefore,

$$\frac{d}{ds}(\vec{r} \cdot \hat{b}) = 0,$$

i.e., $\vec{r} \cdot \hat{b} = \text{constant}$.

Again, $\hat{b}$ is a constant vector of unit magnitude; hence $\vec{r} \cdot \hat{b}$ is the projection of the position vector $\vec{r}$ on $\hat{b}$ and is same at all points of the curve. This implies that the curve must lie in a plane.

Screw-Curvature

The arc rate at which the principal normal changes direction $\left(i.e., \left|\frac{d\hat{n}}{ds}\right|\right)$ is called the *screw curvature vector* and its magnitude is given by

$$\sqrt{\kappa^2 + \tau^2}.$$

Serret–Frenet Formulae:

The following set of three relations involving space derivatives of fundamental unit vectors $\hat{t}, \hat{n}, \hat{b}$ are known as Serret–Frenet Formulae:

1. $\dfrac{d\hat{t}}{ds} = \kappa\hat{n}$ 2. $\dfrac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}$ 3. $\dfrac{d\hat{b}}{ds} = -\tau\hat{n}$

Serret–Frenet formulae can be represented in matrix form as

$$\begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}$$

Curvature and Torsion of a Curve:

For a curve $\vec{r} = \vec{r}(t)$,

$$\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}, \qquad \tau = \frac{[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

We know that,

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{t}\frac{ds}{dt} \tag{1}$$

$$\therefore |\dot{\vec{r}}| = \frac{ds}{dt} = \dot{s} \tag{2}$$

Differentiating again,

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\hat{t}\frac{ds}{dt}\right)$$
$$= \frac{d\hat{t}}{ds}\left(\frac{ds}{dt}\right)^2 + \hat{t}\frac{d^2s}{dt^2}$$

Since $\frac{d\hat{t}}{ds} = \kappa\hat{n}$, we get

$$\ddot{\vec{r}} = \kappa\hat{n}\left(\frac{ds}{dt}\right)^2 + \hat{t}\frac{d^2s}{dt^2} \tag{3}$$

Taking the cross product of $\dot{\vec{r}}$ and $\ddot{\vec{r}}$,

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \kappa\left(\frac{ds}{dt}\right)^3 \hat{b}$$

Hence,

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \kappa\left(\frac{ds}{dt}\right)^3 \tag{4}$$

Differentiating, we get

$$\dot{\vec{r}} \times \dddot{\vec{r}} + \ddot{\vec{r}} \times \ddot{\vec{r}} = \dot{s}^3\kappa\dot{s}\,\dot{\hat{b}} + \hat{b}\,\frac{d}{dt}\left(\dot{s}^3\kappa\right) \qquad [\because \dot{\hat{b}} = -\tau\hat{n}] \tag{5}$$

Again taking the scalar product of (3) and (5), we get

$$[\dot{\vec{r}},\ \ddot{\vec{r}},\ \dddot{\vec{r}}] = -\dot{s}^{\,6}\kappa^2\tau \tag{6}$$

Also from (2) and (4), we have

$$\begin{aligned} \dot{s}^{\,3}\kappa|\hat{b}| &= |\dot{\vec{r}} \times \ddot{\vec{r}}| \\ \Rightarrow |\dot{\vec{r}}|^3\kappa|\hat{\vec{b}}| &= |\dot{\vec{r}} \times \ddot{\vec{r}}| \\ \Rightarrow \kappa &= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \qquad [\because |\hat{b}| = 1] \end{aligned}$$

From (6) and (4), we have

$$\tau = \frac{[\dot{\vec{r}},\ \ddot{\vec{r}},\ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

Q: Show that

$$\kappa = |\vec{r}' \times \vec{r}''| \quad \text{and} \quad \tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{|\vec{r}' \times \vec{r}''|^2}$$

Proof: We know that $\vec{r}' = \hat{t}$ and $\vec{r}'' = \kappa\hat{n}$

$$\therefore \quad \vec{r}' \times \vec{r}'' = \hat{t} \times \kappa\hat{n} = \kappa\hat{b}$$

or,

$$|\vec{r}' \times \vec{r}''| = \kappa|\hat{b}| = \kappa \tag{1}$$

Again,

$$\vec{r}' = \hat{t} = 1.\,\hat{t} + 0.\,\hat{n} + 0.\,\hat{b} \tag{2}$$

$$\vec{r}'' = \kappa\hat{n} = 0.\,\hat{t} + \kappa\hat{n} + 0.\,\hat{b} \tag{3}$$

$$\vec{r}''' = \kappa\frac{d\hat{n}}{ds} + \frac{d\kappa}{ds}\hat{n} \tag{2}$$

$$= \kappa(\tau\hat{b} - \kappa\hat{t}) + \kappa'\hat{n} \tag{3}$$

$$= -\kappa^2\hat{t} + \kappa'\hat{n} + \kappa\tau\hat{b} \tag{4}$$

From (2), (3) and (4), we have

$$[\vec{r}', \vec{r}'', \vec{r}'''] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \end{vmatrix}$$

$$= \kappa^2\tau \tag{5}$$

or,

$$\tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{\kappa^2} = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{|\vec{r}' \times \vec{r}''|^2} \qquad [\because \kappa = |\vec{r}' \times \vec{r}''|]$$

Theorem: Show that the necessary and sufficient condition for the curve to be a plane curve is $[\vec{r}', \vec{r}'', \vec{r}'''] = 0$.

Proof: We have, $[\vec{r}', \vec{r}'', \vec{r}'''] = \kappa^2\tau$.
In case $[\vec{r}', \vec{r}'', \vec{r}'''] = 0$, then either $\kappa = 0$ or $\tau = 0$.
Let, $\tau \neq 0$ at some point of the curve then in the neighbourhood of this point $\tau \neq 0$, therefore $\kappa = 0$ in the neighbourhood of this point.
Hence the arc is a straight line and therefore $\tau = 0$ on this line which contradicts our hypothesis. Thus $\tau = 0$ at all points and the curve is a plane.

Conversely if $\tau = 0$ i.e., the curve is a plane curve then $[\vec{r}', \vec{r}'', \vec{r}'''] = 0$.
Therefore the condition is necessary as well as sufficient.

THANK YOU

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