Space Curves - Differential Geometry

Presented by

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Osculating Plane:

Curve of Class ≥ 2:

A curve is said to be of class ≥ 2 if it is sufficiently smooth. Let the curve be represented by its position vector $\vec{r}(t)$.

Then,

- the first derivative $\vec{r}\,'(t)$ (velocity),
- the second derivative $\vec{r}\,''(t)$ (acceleration),

both exist and are continuous.

In simple terms, a curve of class ≥ 2 has a well-defined tangent vector and a well-defined curvature at every point.

Tangent Line to a Space Curve:

Let us consider a point P on a space curve. The tangent line at the point P is the line that

- touches the curve at P

- has the same direction as the curve at that point

Mathematically, if $P = \vec{r}(t_0)$, then the direction of the tangent line at P is given by the vector $\vec{r}\,'(t_0)$.

Hence, the tangent line at the point P is uniquely determined.

A Nearby Point and the Associated Plane:

Let Q be another point on the curve, chosen very close to the point P.

The tangent line at P together with the point Q determines a unique plane. This is because a line and a point not lying on that line uniquely define a plane.

Thus,

$$\text{Tangent line at } P + \text{ nearby point } Q \implies \text{a plane}$$

This plane depends on the position of the point Q.

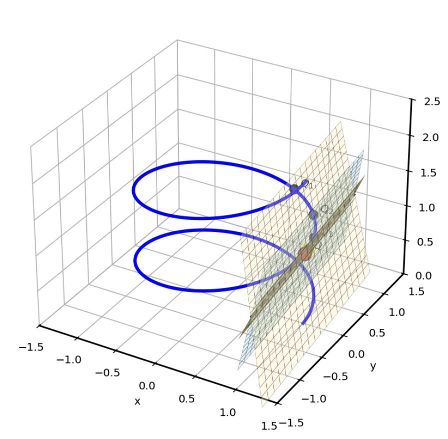
Limiting Position of the Plane:

As the point Q approaches P (that is, $Q \to P$), the corresponding plane changes continuously.

However, in the limit, the plane approaches a fixed position.
This limiting plane is called the *osculating plane* at the point P. The osculating plane is the plane that best fits the curve near the point P.

Definition

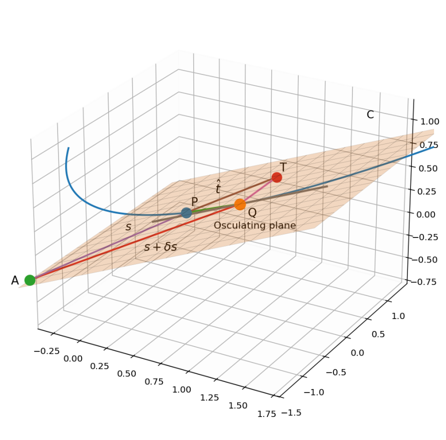
The ***osculating plane*** at a point P of a curve of class ≥ 2 is the limiting position of the plane which contains the tangent line at P and a neighbouring point Q on the curve as $Q \to P$.
Alternatively, if P, Q, R are three consecutive points on the curve, then the limiting position of the plane PQR, as the points Q and R tend to P, is called the ***osculating plane*** at the point P.



Planes through Tangent at P and Q_1, Q_2, Q_3 $(Q \to P)$

Equation of the Osculating Plane

To find the equation of the osculating plane at a given point P of the curve C.



Let $\vec{r} = \vec{r}(s)$ be the given curve of class ≥ 2 with respect to the parameter s, the arc length.

Let the arc length be measured from some point, say A, such that arc $AP = s$, arc $AQ = s + \delta s$, so that arc $PQ = \delta s$.

The position vector of the point P can be taken as $\vec{r}(s)$. The position vector of the point Q can be taken as $\vec{r}(s + \delta s)$.

Let R be the position vector of the current point T on the plane containing the tangent line at P and the point Q.

The unit tangent vector at P is

$$\hat{\mathbf{t}} = \mathbf{r}'(s).$$

Now,

$$\overrightarrow{PT} = \mathbf{R} - \mathbf{r}(s), \qquad \hat{\mathbf{t}} = \mathbf{r}'(s), \qquad \overrightarrow{PQ} = \mathbf{r}(s + \delta s) - \mathbf{r}(s),$$

and these vectors lie in the plane TPQ.

Hence, their scalar triple product must be zero, i.e.,

$$\left[\mathbf{R} - \mathbf{r}(s),\ \mathbf{r}'(s),\ \mathbf{r}(s + \delta s) - \mathbf{r}(s)\right] = 0. \tag{1}$$

Equation (1) is the equation of the plane TPQ.
Now, expanding $\mathbf{r}(s + \delta s)$ in powers of δs by Taylor's theorem, we have

$$\mathbf{r}(s + \delta s) = \mathbf{r}(s) + \delta s\,\mathbf{r}'(s) + \frac{(\delta s)^2}{2!}\,\mathbf{r}''(s) + O(\delta s^3). \tag{2}$$

Putting the value of $\mathbf{r}(s + \delta s)$ from (2) into (1), we get

$$\left[\mathbf{R} - \mathbf{r}(s),\ \mathbf{r}'(s),\ \delta s\,\mathbf{r}'(s) + \frac{(\delta s)^2}{2!}\,\mathbf{r}''(s) + O(\delta s^3)\right] = 0.$$

Since $\mathbf{r}'(s) \times \mathbf{r}'(s) = \mathbf{0}$, this reduces to

$$\left[\mathbf{R} - \mathbf{r}(s),\ \mathbf{r}'(s),\ \mathbf{r}''(s) + O(\delta s^3)\right] = 0.$$

Hence, the limiting position of the plane as $Q \to P$, i.e. as $\delta s \to 0$, is

$$\big[\mathbf{R} - \mathbf{r}(s),\ \mathbf{r}'(s),\ \mathbf{r}''(s)\big] = 0. \tag{3}$$

Provided the vectors $\mathbf{r}'(s)$ and $\mathbf{r}''(s)$ are linearly independent, equation (3) can be written as

$$\big[\mathbf{R} - \mathbf{r}(s),\ \mathbf{r}'(s),\ \mathbf{r}''(s)\big] = 0, \tag{4}$$

which is the equation of the osculating plane at P.

Remark:
The vectors $\vec{r}'(s)$ and $\vec{r}''(s)$ lie in the osculating plane and therefore the vector $\vec{r}'(s) \times \vec{r}''(s)$ is normal to the osculating plane at P.

Deduction 1. Osculating plane at a point of inflexion.

A point P where $\vec{r}'' = 0$ is called a point of inflexion, and the tangent line at P is called the inflexional tangent.

For finding the equation of the osculating plane at a point of inflexion, it will be shown that when a curve is analytic, there exists a definite osculating plane at a point of inflexion P provided the curve is not a straight line.

Since $\vec{r}'$ is a vector of constant magnitude unity, it is perpendicular to its derivative $\vec{r}''$ so that

$$\vec{r}' \cdot \vec{r}'' = 0$$

Differentiating this, we get

$$\vec{r}' \cdot \vec{r}''' + \vec{r}'' \cdot \vec{r}'' = 0 \tag{5}$$

Again P is a point of inflexion, $\vec{r}'' = 0$. Hence (5) reduces to

$$\vec{r}' \cdot \vec{r}''' = 0$$

This shows that $\vec{r}'$ is linearly independent of $\vec{r}'''$ except when $\vec{r}''' = 0$. Continuing this argument, we shall arrive at the result

$$\vec{r}' \cdot \vec{r}^{(k)} = 0,$$

where $\vec{r}^{(k)}$ $(k \geq 2)$ is the first non-zero derivative of $\vec{r}$ at P. We then have

$$\vec{r}(s + \delta s) - \vec{r}(s) = \frac{(\delta s)^k}{k!}\,\vec{r}^{(k)}(s) + O(\delta s^{k+1})$$

Hence the equation of the osculating plane at P is

$$[\vec{R} - \vec{r}(s),\, \vec{r}'(s),\, \vec{r}^{(k)}(s)] = 0$$

Remark:
In case $\vec{r}^{(k)} = 0$ for all $k \geq 2$, then since the curve is analytic, we see that $\vec{r}'$ is constant and the curve is a straight line.

Deduction 2. Osculating plane for a curve with general parameter t.

If the equations of a curve are given in terms of a general parameter t, then

$$\vec{r}' = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\,\frac{dt}{ds} = \frac{\dot{\vec{r}}}{\dot{s}},$$

$$\vec{r}'' = \frac{d\vec{r}'}{ds} = \frac{\ddot{\vec{r}}}{\dot{s}^{\,2}} - \frac{\dot{\vec{r}}\,\ddot{s}}{\dot{s}^{\,3}}.$$

Putting these values of $\vec{r}'$ and $\vec{r}''$ in (4), we get

$$(\vec{R} - \vec{r}) \cdot \left(\frac{\dot{\vec{r}}}{\dot{s}} \times \left(\frac{\ddot{\vec{r}}}{\dot{s}^{\,2}} - \frac{\dot{\vec{r}}\,\ddot{s}}{\dot{s}^{\,3}} \right) \right) = 0$$

$$\Rightarrow (\vec{R} - \vec{r}) \cdot (\dot{\vec{r}} \times \ddot{\vec{r}}) = 0 \qquad (\because\ \dot{\vec{r}} \times \dot{\vec{r}} = 0)$$

$$\Rightarrow [\,\vec{R} - \vec{r},\ \dot{\vec{r}},\ \ddot{\vec{r}}\,] = 0.$$

Deduction 3. Cartesian form.

Let (X, Y, Z) be the coordinates of a current point T on the osculating plane at P, the coordinates of point P being (x, y, z). Then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} \quad \text{and} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore\ \vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$$

Again,

$$\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}, \qquad \ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

Hence,

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Ex.1. Find the osculating plane at the point t on the helix

$$\vec{r} = (a\cos t,\ a\sin t,\ ct)$$

Solution: The equations of the helix are

$$x = a\cos t, \qquad y = a\sin t, \qquad z = ct$$

Differentiating with respect to t, we get

$$\dot{x} = -a\sin t, \qquad \dot{y} = a\cos t, \qquad \dot{z} = c,$$

$$\ddot{x} = -a\cos t, \qquad \ddot{y} = -a\sin t, \qquad \ddot{z} = 0$$

Therefore, the equation of the osculating plane at the point t is

$$\begin{vmatrix} X - a\cos t & Y - a\sin t & Z - ct \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = 0.$$

Expanding the determinant, we get

$$(Z - ct)\Big[(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)\Big]$$
$$- c\Big[(X - a\cos t)(-a\sin t) - (Y - a\sin t)(-a\cos t)\Big] = 0$$

Or,

$$c(X\sin t - Y\cos t - at) + az = 0.$$

Thus, the angle between the tangent to the curve and the given line is independent of t and remains constant for every point on the curve.

Ex.2. For the curve $\vec{r} = (3t,\ 3t^2,\ 2t^3)$, show that any plane meets it in three points and deduce the equation of the osculating plane at $t = t_1$.

Solution: The equations of the curve are

$$x = 3t, \qquad y = 3t^2, \qquad z = 2t^3 \tag{1}$$

The most general equation of a plane is

$$Ax + By + Cz + D = 0 \tag{2}$$

Putting the values of x, y, z from (1) into (2), we get

$$F(t) = A(3t) + B(3t^2) + C(2t^3) + D = 0 \tag{3}$$

The equation (3) is cubic in t, and hence any plane meets the curve in three points. The osculating plane has three-point contact with the curve at $t = t_1$; thus

$$F(t_1) = 0, \qquad F'(t_1) = 0, \qquad F''(t_1) = 0$$

These conditions give

$$3At_1 + 3Bt_1^2 + 2Ct_1^3 + D = 0, \tag{4}$$

$$3A + 6Bt_1 + 6Ct_1^2 = 0, \tag{5}$$

$$6B + 12Ct_1 = 0 \tag{6}$$

Eliminating A, B, C, D from (2), (4), (5) and (6), the equation of the osculating plane is

$$\begin{vmatrix} x & y & z & 1 \\ 3t_1 & 3t_1^2 & 2t_1^3 & 1 \\ 3 & 6t_1 & 6t_1^2 & 0 \\ 0 & 6 & 12t_1 & 0 \end{vmatrix} = 0$$

or equivalently,

$$\begin{vmatrix} x - 3t_1 & y - 3t_1^2 & z - 2t_1^3 \\ 1 & 2t_1 & 2t_1^2 \\ 0 & 1 & 2t_1 \end{vmatrix} = 0$$

Expanding, we get

$$2t_1^2(x - 3t_1) - 2t_1(y - 3t_1^2) + (z - 2t_1^3) = 0,$$

or

$$2t_1^2 x - 2t_1 y + z = 2t_1^3$$

THANK YOU

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