Space Curves - Differential Geometry

Presented by

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Order of Contact Between Curves and Surfaces:

Let us consider a curve C and surface S given by the following equations

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{1}$$

$$F(x, y, z) = 0 \tag{2}$$

The values of t corresponding to the points which are common to C and S are given by the solution of the equation obtained from (1) and (2) on eliminating x, y, z, i.e., by

$$F[f(t), g(t), h(t)] = 0 \quad \text{or} \quad F(t) = 0 \tag{3}$$

Let t_0 be one of the solutions of (3), then

$$F(t_0) = 0$$

Now expanding $F(t)$ about t_0 by Taylor's theorem in powers of $(t - t_0)$, we get

$$F(t) = F(t_0) + (t - t_0)F'(t_0) + \frac{(t - t_0)^2}{2!}F''(t_0) + \cdots + \frac{(t - t_0)^n}{n!}F^{(n)}(t_0) + \cdots$$

or,

$$F(t) = (t - t_0)F'(t_0) + \frac{(t - t_0)^2}{2!}F''(t_0) + \cdots + \frac{(t - t_0)^n}{n!}F^{(n)}(t_0) + \cdots \qquad [\because F(t_0) = 0]$$

Now the following different cases arise:

1. If $F'(t_0) \neq 0$, then t_0 is a **simple zero** of $F(t)$ and in this case C and S are said to have **simple intersection**.

2. If $F'(t_0) = 0$ but $F''(t_0) \neq 0$, then t_0 is a **double zero** of $F(t)$ and the curve C and surface S have **two-point contact** or **contact of first order**.
3. If $F'(t_0) = F''(t_0) = 0$ but $F'''(t_0) \neq 0$, then t_0 is a **triple zero** of $F(t)$ and C and S have **three-point contact** or **contact of second order**.

In general, if

$$F'(t_0) = F''(t_0) = \cdots = F^{(r)}(t_0) = 0, \quad \text{but} \quad F^{(r+1)}(t_0) \neq 0,$$

then C and S are said to have $(r+1)$**-point contact** or **contact of** r**th order**.

Ex.1. Prove that if the circle

$$lx + my + nz = 0, \qquad x^2 + y^2 + z^2 = 2cz$$

has three point contact at the origin with the paraboloid

$$ax^2 + by^2 = 2z,$$

then

$$c = \frac{l^2 + m^2}{bl^2 + am^2}.$$

Solution: Equation of the circle are

$$lx + my + nz = 0 \tag{1}$$

$$x^2 + y^2 + z^2 = 2cz \tag{2}$$

Differentiating these equations w.r.t. the parameter t, we get

$$l\dot{x} + m\dot{y} + n\dot{z} = 0 \tag{3}$$

$$x\dot{x} + y\dot{y} + z\dot{z} = c\dot{z} \tag{4}$$

At the origin $x = 0, y = 0, z = 0$, the equations (3) and (4) reduce to

$$l\dot{x} + m\dot{y} = 0 \quad \text{and} \quad \dot{z} = 0$$

$$\frac{\dot{x}}{m} = -\frac{\dot{y}}{l} = \lambda \text{ (say)} \tag{5}$$

Differentiating (4) again, we get

$$\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} + \dot{z}^2 + z\ddot{z} = c\ddot{z}$$

At the origin this reduces to

$$\dot{x}^2 + \dot{y}^2 = c\ddot{z} \tag{6}$$

Again, equation of the paraboloid is

$$ax^2 + by^2 - 2z = 0$$

In order that the paraboloid may have a three point contact with the circle, we must have

$$2a\dot{x}^2 + 2b\dot{y}^2 - 2\dot{z} = 0 \tag{7}$$

and

$$2a\dot{x}^2 + 2ax\ddot{x} + 2b\dot{y}^2 + 2by\ddot{y} - 2\ddot{z} = 0 \tag{8}$$

At the origin equation (8) reduces to

$$a\dot{x}^2 + b\dot{y}^2 = \ddot{z} \tag{9}$$

From (6) and (9) on eliminating $\ddot{z}$, we get

$$c = \frac{\dot{x}^2 + \dot{y}^2}{a\dot{x}^2 + b\dot{y}^2} = \frac{m^2 + l^2}{am^2 + bl^2}, \qquad [\text{From } (5),\ \dot{x} = m\lambda,\ \dot{y} = -l\lambda]$$

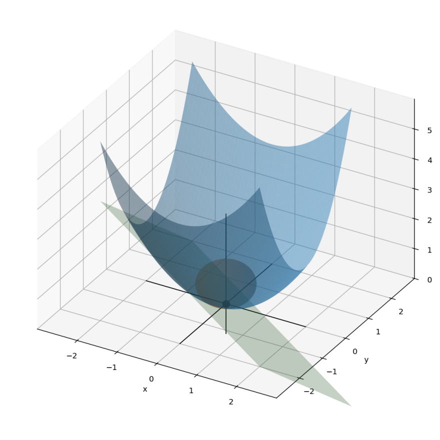


Figure 1: Three-point contact of Sphere, Paraboloid and Plane at the Origin

Ex.2. Find the equation of the plane that has three point contact at the origin with the curve

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1.$$

Solution: The equation of any plane through the origin is

$$ax + by + cz = 0 \tag{1}$$

The equations of the given curve are

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1 \tag{2}$$

Eliminating x, y, z from (1) and (2), we obtain

$$F(t) = a(t^4 - 1) + b(t^3 - 1) + c(t^2 - 1) = 0$$

Therefore,

$$F'(t) = 4at^3 + 3bt^2 + 2ct,$$

$$F''(t) = 12at^2 + 6bt + 2c$$

The origin means

$$(x, y, z) = (0, 0, 0)$$

Setting each coordinate equal to zero, we get

$$t^4 - 1 = 0 \Rightarrow t^4 = 1$$

$$t^3 - 1 = 0 \Rightarrow t^3 = 1$$

$$t^2 - 1 = 0 \Rightarrow t^2 = 1$$

The only common solution of all three equations is $t = 1$.
Since the plane has three point contact with the curve at the origin, i.e., at $t = 1$, we have

$$F'(1) = 4a + 3b + 2c = 0,$$

$$F''(1) = 12a + 6b + 2c = 0$$

Solving these equations, we get

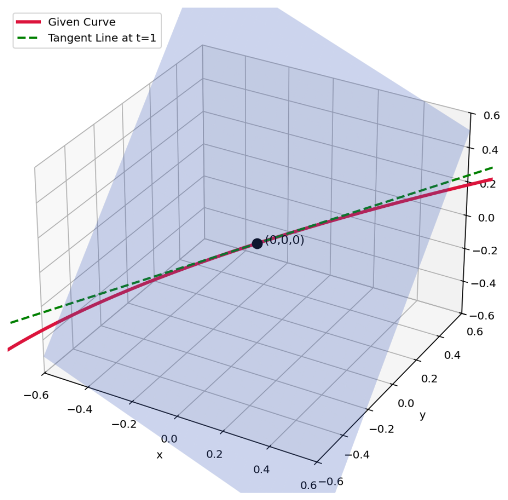
$$\frac{a}{6-12} = \frac{b}{24-8} = \frac{c}{24-36},$$

or

$$\frac{a}{3} = \frac{b}{-8} = \frac{c}{6}$$

Hence, the required plane is

$$\boxed{3x - 8y + 6z = 0}$$



Three-Point Contact of Curve with Plane $3x - 8y + 6z = 0$

Ex.3. Show that the condition that four consecutive points of a curve should be coplanar is

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

Solution: Let us take parametric equations of the curve as follows:

$$x = f(t), \quad y = g(t), \quad z = h(t). \tag{1}$$

Again, the equation of the plane through a point (x, y, z) on the curve is given by

$$(X - x)l + (Y - y)m + (Z - z)n = 0. \tag{2}$$

The plane (2) will pass through four consecutive points if it has three-point contact with the curve, i.e.,

$$F(t_0) = F'(t_0) = F''(t_0) = F'''(t_0) = 0$$

The equation of the plane through the point corresponding to $t = t_0$ is

$$(X - x(t_0))l + (Y - y(t_0))m + (Z - z(t_0))n = 0$$

Substituting a general point of the curve

$$X = x(t), \quad Y = y(t), \quad Z = z(t),$$

we define

$$F(t) = [x(t) - x(t_0)]l + [y(t) - y(t_0)]m + [z(t) - z(t_0)]n$$

Putting $t = t_0$, we get

$$F(t_0) = [x(t_0) - x(t_0)]l + [y(t_0) - y(t_0)]m + [z(t_0) - z(t_0)]n = 0$$

Thus the plane passes through the point corresponding to t_0.

Differentiating $F(t)$ with respect to t,

$$F'(t) = x'(t)l + y'(t)m + z'(t)n$$

Evaluating at $t = t_0$,

$$F'(t_0) = x'l + y'm + z'n = 0 \tag{3}$$

Differentiating again,

$$F''(t) = x''(t)l + y''(t)m + z''(t)n$$

At $t = t_0$,

$$F''(t_0) = x''l + y''m + z''n = 0 \tag{4}$$

Differentiating once more,

$$F'''(t) = x'''(t)l + y'''(t)m + z'''(t)n$$

At $t = t_0$,

$$F'''(t_0) = x'''l + y'''m + z'''n = 0 \tag{5}$$

From the conditions

$$F(t_0) = F'(t_0) = F''(t_0) = F'''(t_0) = 0,$$

Thus from equations (3), (4), and (5) we have,

$$x'l + y'm + z'n = 0 \tag{6}$$
$$x''l + y''m + z''n = 0 \tag{7}$$
$$x'''l + y'''m + z'''n = 0. \tag{8}$$

Here, dashes denote differentiation with respect to t.

Now, eliminating l, m, n from equations (6), (7), and (8), we obtain

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

Hence proved.

THANK YOU

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