Space Curves - Differential Geometry

Presented by

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Space Curves

Space Curve: A **space curve** is a curve that lies in three-dimensional space and is represented by using a vector-valued function

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k},$$

where

- t is a parameter (often time or any real variable).
- $x(t)$, $y(t)$, and $z(t)$ are coordinate functions.

e.g.

Helix: $\vec{r}(t) = (\cos t, \sin t, t)$

Straight line: $\vec{r}(t) = \vec{a} + t\,\vec{b}$

Circle in space: $\vec{r}(t) = (a\cos t, a\sin t, 0)$

Space Curves

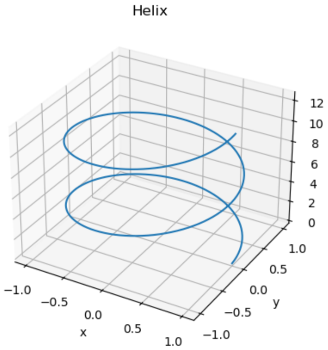


Figure 1: Helix

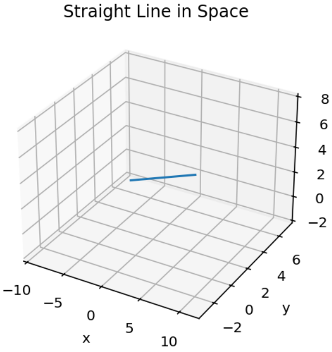


Figure 2: Straight Line

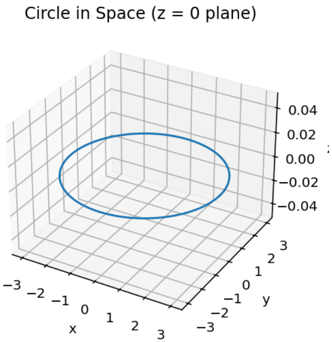


Figure 3: Circle

Space Curves

- The curve is known as *plane curve* if it lies on a plane, otherwise it is said to be a *skew twisted* or *tortuous curve*.
- Unlike a plane curve (2D), a space curve moves freely in three-dimensional space.
- The parametric equations of a curve are: $x = x(t), \quad y = y(t), \quad z = z(t)$, where x, y, z are real valued functions of a single real parameter t ranging over a set of values $a \leq t \leq b$.
- A space curve may also be given as the intersection of two surfaces whose equations are

$$f_1(x, y, z) = 0 \quad \text{and} \quad f_2(x, y, z) = 0, \quad \text{then}$$

$$f_1(x, y, z) = 0; \; f_2(x, y, z) = 0$$

Space Curves

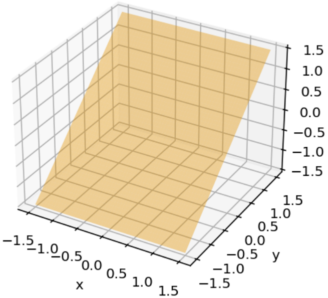


Figure 4: Plane: $z = y$

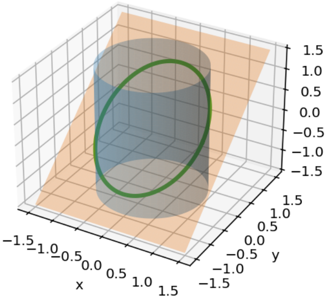


Figure 5: Intersection of two surfaces

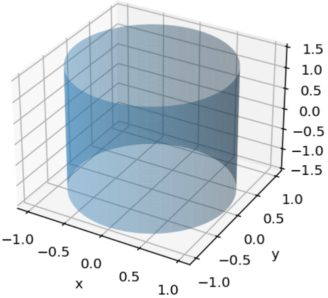


Figure 6: Cylinder: $x^2 + y^2 = 1$

Arc Length of a Space Curve: The **arc length** is the actual length of the curve between two points. For a space curve $\vec{r}(t)$, the arc length from $t = a$ to $t = b$ is given by

$$S = \int_a^b \left| \frac{d\vec{r}}{dt} \right| dt$$

where

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Equivalently,

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The arc length is obtained by adding infinitely small straight-line segments along the curve to determine its total length.

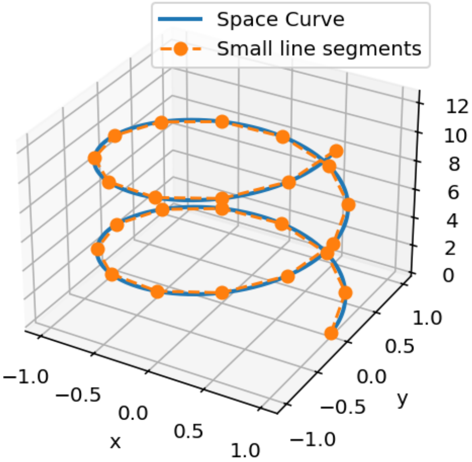


Figure 7: Arc Length as sum of small line segments

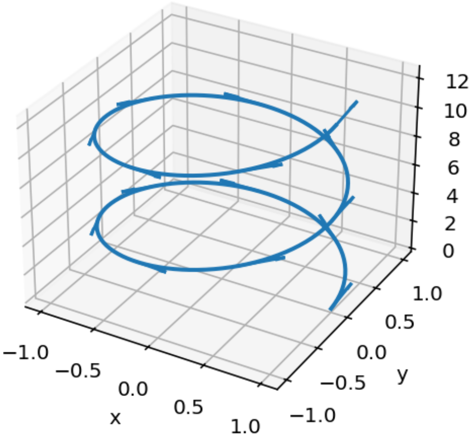


Figure 8: Tangent vectors along a space curve

Tangent Vector: A **tangent vector** to a space curve $\vec{r}(t)$ at a point is the derivative

$$\frac{d\vec{r}}{dt}$$

representing the instantaneous direction of the curve in three-dimensional space.

Velocity Vector: The **velocity vector** is defined as

$$\vec{v} = \frac{d\vec{r}}{dt}$$

This vector is tangent to the curve but is not necessarily of unit length.

Unit Tangent Vector: The **unit tangent vector** is given by

$$\vec{T} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|}$$

Alternatively, the unit tangent vector can be written as

$$\vec{T} = \frac{d\vec{r}}{ds}$$

where s denotes the arc length parameter.
If a particle moves along the curve, the tangent gives the direction of velocity.

Normal Vector: The **normal vector** shows the direction in which the curve is bending. It is defined as the direction of change of the tangent vector.

Unit normal vector: The **unit normal vector** is

$$\hat{\mathbf{N}} = \frac{\dfrac{d\hat{\mathbf{T}}}{ds}}{\left|\dfrac{d\hat{\mathbf{T}}}{ds}\right|}, \quad \text{where } s \text{ is the arc length}$$

- Tangent → direction of motion
- Normal → direction of curvature (how the path is turning)

Space Curves

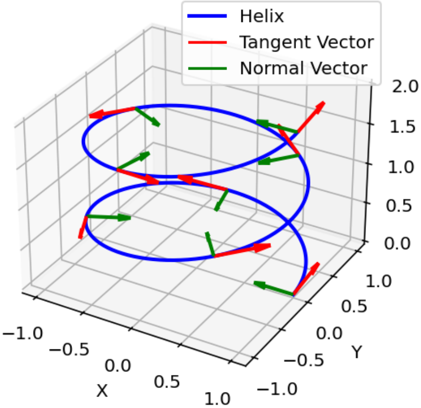


Figure 9: Tangent vector and Normal vector

To find the unit vector along the tangent to a given curve

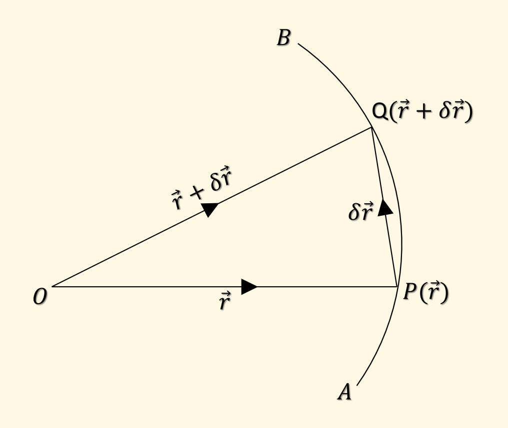
Let $P(\vec{r})$ and $Q(\vec{r}+\delta\vec{r})$ be two consecutive points on a curve C. The position vectors of P and Q with respect to the origin O are respectively $\vec{r}$ and $\vec{r}+\delta\vec{r}$, i.e.,

$$\overrightarrow{OP}=\vec{r}, \qquad \overrightarrow{OQ}=\vec{r}+\delta\vec{r}$$

$$\therefore\ \overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\delta\vec{r}$$

Let point A on the curve be a fixed point and let arc $AP=s$ and arc $AQ=s+\delta s$.

$$\therefore\ \text{arc } PQ=\delta s$$



Now unit vector along the chord PQ is

$$\widehat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\delta\vec{r}}{\text{chord } PQ} = \left(\frac{\delta\vec{r}}{\delta s}\right)\left(\frac{\delta s}{\text{chord } PQ}\right) = \left(\frac{\delta\vec{r}}{\delta s}\right)\left(\frac{\text{arc } PQ}{\text{chord } PQ}\right)$$

Now when point Q tends to point P, the chord PQ tends to the tangent at P. Hence the unit vector along the tangent at P is

$$\lim_{Q\to P}\left(\frac{\delta\vec{r}}{\delta s}\right)\left(\frac{\text{arc } PQ}{\text{chord } PQ}\right) = \frac{d\vec{r}}{ds} = \vec{r}'$$

The symbol $\hat{t}$ is used for the unit vector along the tangent at P and is taken positive in the direction of increasing s, i.e.,

$$\hat{t} = \vec{r}'$$

If $\vec{r} = (x, y, z)$, where x, y, z are components of $\vec{r}$, then

$$\frac{d\vec{r}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \hat{t}$$

Here $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are components of $\hat{t}$.
Thus,

$$\hat{t} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\,\hat{\mathbf{i}} + \frac{dy}{ds}\,\hat{\mathbf{j}} + \frac{dz}{ds}\,\hat{\mathbf{k}} \tag{1}$$

Since the magnitude of $\vec{t}$ is unity, on squaring (1) we get

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$$

or

$$1 = \left(\frac{dx}{dt}\right)^2 \left(\frac{dt}{ds}\right)^2 + \left(\frac{dy}{dt}\right)^2 \left(\frac{dt}{ds}\right)^2 + \left(\frac{dz}{dt}\right)^2 \left(\frac{dt}{ds}\right)^2$$

or

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left|\frac{d\vec{r}}{dt}\right|^2 \tag{2}$$

where t is any parameter.
The formula (2) may be used to find the arc length. Thus

$$s = \int_{t_0}^{t} \left|\dot{\vec{r}}\right| dt = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt \tag{3}$$

Note. Two points P and Q on a curve are consecutive, it does not mean that the points are near in space but the values of the parameter differ by an indefinitely small amount.

Tangent Line at a Point on a Space Curve

To find the equation of the tangent line at a given point $P(\vec{r})$ on the curve C.

Let $\vec{R}$ represent the position vector of a current point on the tangent at P, $\vec{t}$ is unit vector along tangent at P, thus the equation of tangent line at P is

$$\vec{R} = \vec{r} + \lambda \vec{t}$$
$$\text{or,} \quad \vec{R} = \vec{r} + \lambda \vec{r}' \tag{1}$$

where λ is the scalar parameter.

If instead of the parameter s any other parameter t is used then since

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{t}\,\frac{ds}{dt}$$

is parallel to the vector $\hat{t}$, the equation of tangent line may be written as

$$\vec{R} = \vec{r} + \lambda\dot{\vec{r}}$$

Cor 1. Cartesian Representation

Let

$$\vec{R} = (X, Y, Z), \quad \vec{r} = (x, y, z), \quad \vec{r}' = (x', y', z')$$

The tangent line at the point (x, y, z) is

$$\vec{R} = \vec{r} + \lambda\vec{r}'$$

$$\text{or,} \quad X\hat{\mathbf{i}} + Y\hat{\mathbf{j}} + Z\hat{\mathbf{k}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} + \lambda(x'\hat{\mathbf{i}} + y'\hat{\mathbf{j}} + z'\hat{\mathbf{k}}) \tag{1}$$

Tangent Line at a Point on a Space Curve

or, whence

$$\frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'}=\lambda$$

The direction cosines of the tangent are therefore x', y', z'.
If the curve is given by

$$f_1(x,y,z)=0 \quad \text{and} \quad f_2(x,y,z)=0,$$

then we have

$$\frac{\partial f_1}{\partial x}x'+\frac{\partial f_1}{\partial y}y'+\frac{\partial f_1}{\partial z}z'=0,$$

$$\frac{\partial f_2}{\partial x}x'+\frac{\partial f_2}{\partial y}y'+\frac{\partial f_2}{\partial z}z'=0.$$

Tangent Line at a Point on a Space Curve

Solving these equations, we get

$$\frac{x'}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z}-\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}}=\frac{y'}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}}=\frac{z'}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}}.$$

Putting the values of x', y', z' in (1), we get the equation of the tangent.

Cor. 2. In case equation of the curve is given by

$$f_1(x,y,z)=0, \quad f_2(x,y,z)=0,$$

we have

$$\frac{\partial f_1}{\partial x}\dot{x}+\frac{\partial f_1}{\partial y}\dot{y}+\frac{\partial f_1}{\partial z}\dot{z}=0,$$

Tangent Line at a Point on a Space Curve

$$\frac{\partial f_2}{\partial x}\dot{x} + \frac{\partial f_2}{\partial y}\dot{y} + \frac{\partial f_2}{\partial z}\dot{z} = 0.$$

On solving these equations, we get

$$\frac{\dot{x}}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}} = \frac{\dot{y}}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}} = \frac{\dot{z}}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}}.$$

Above equation gives the direction cosines of the tangent to the curve of intersection of two surfaces.
The Cartesian equivalent of $\vec{R} = \vec{r} + \lambda \vec{r}'$ is

$$\frac{X - x}{\dot{x}} = \frac{Y - y}{\dot{y}} = \frac{Z - z}{\dot{z}} = c \tag{1}$$

Tangent Line at a Point on a Space Curve

Thus putting the values of $\dot{x}, \dot{y}, \dot{z}$ in (1), we get the equation of the tangent line.

Cor. 3. In case at a point P, $\mathbf{r}'' = 0$, the tangent line at P is called *inflexional* and the point P is called the *point of inflexion*.

Ex.1. Show that the tangent at any point of the curve whose equations, referred to rectangular axes, are $x = 3t, \quad y = 3t^2, \quad z = 2t^3$ makes a constant angle with the line $y = z - x = 0$.

Solution: For a parametric curve

$$\vec{r}(t) = \big(x(t),\, y(t),\, z(t)\big),$$

the tangent vector is

$$\vec{r}\,'(t) = \big(\dot{x},\, \dot{y},\, \dot{z}\big)$$

Here,

$$\dot{x} = 3, \qquad \dot{y} = 6t, \qquad \dot{z} = 6t^2$$

Hence, a tangent direction vector is

$$\vec{T} = (3,\, 6t,\, 6t^2)$$

Since direction cosines depend only on ratios, dividing by 3 gives

$$\vec{T} \sim (1,\, 2t,\, 2t^2)$$

i.e., the direction cosines are proportional to $1,\ 2t,\ 2t^2$.
The line is given by

$$y = 0, \qquad z - x = 0$$

Thus,

$$y = 0, \qquad z = x,$$

and any point on the line can be written as $(x,\, 0,\, x)$.
Hence, a direction vector of the line is

$$\vec{L} = (1,\, 0,\, 1)$$

Its magnitude is

$$|\vec{L}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

Therefore, the direction cosines of the line are

$$\left(\frac{1}{\sqrt{2}},\, 0,\, \frac{1}{\sqrt{2}}\right).$$

The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

Here,

$$\vec{T} = (1,\, 2t,\, 2t^2), \qquad \vec{L} = (1,\, 0,\, 1).$$

Their dot product is

$$\vec{T}\cdot\vec{L} = 1\cdot 1 + 2t\cdot 0 + 2t^2\cdot 1 = 1 + 2t^2$$

The magnitudes are

$$|\vec{T}| = \sqrt{1 + (2t)^2 + (2t^2)^2} = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2,$$

and

$$|\vec{L}| = \sqrt{2}$$

$$\cos\theta = \frac{1 + 2t^2}{(1 + 2t^2)\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

Thus, the angle between the tangent to the curve and the given line is independent of t and remains constant for every point on the curve.

Ex.2. Find the length of the curve given as the intersection of the surfaces

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad x = a\cosh\left(\frac{z}{a}\right)$$

from the point $(a, 0, 0)$ to the point (x, y, z).

Solution: The curve lies on the hyperbolic cylinder

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Using the identity

$$\cosh^2 t - \sinh^2 t = 1,$$

a natural parametrization is

$$x = a\cosh t, \qquad y = b\sinh t$$

From the second surface,

$$x = a\cosh\left(\frac{z}{a}\right),$$

and substituting $x = a\cosh t$, we obtain

$$\cosh t = \cosh\left(\frac{z}{a}\right) \quad \Rightarrow \quad z = at$$

$\therefore$ The parametric equations of the given curve are

$$x = a\cosh t, \qquad y = b\sinh t, \qquad z = at$$

Hence,

$$\dot{x} = a\sinh t, \qquad \dot{y} = b\cosh t, \qquad \dot{z} = a$$

$$\therefore \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{a^2 \sinh^2 t + b^2 \cosh^2 t + a^2}$$
$$= \sqrt{(a^2 + b^2) \cosh^2 t}$$
$$= \sqrt{a^2 + b^2} \, \cosh t$$

At the point $(a, 0, 0)$, we have $t = 0$, and at the point (x, y, z), we have $t = t$. Hence the arc length s is

$$s = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, dt = \int_0^t \sqrt{a^2 + b^2} \, \cosh t \, dt$$

$$\Rightarrow s = \sqrt{a^2 + b^2} \, [\sinh t]_0^t = \sqrt{a^2 + b^2} \, \sinh t$$

Since $\sinh t = \dfrac{y}{b}$, we obtain

$$s = \frac{\sqrt{a^2 + b^2}}{b} \, y \quad \text{which is the required length.}$$

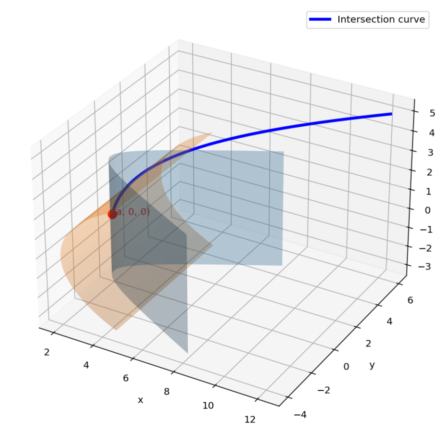


Figure 1: Two Surfaces and Their Intersection Curve

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