1 Analysis of Stress

1.1 The Continuum Concept

The continuum concept is a basic idea used in mathematics and physics to describe things that change smoothly without any breaks or gaps. According to this concept, a quantity can be divided into smaller and smaller parts endlessly, and between any two values there are infinitely many intermediate values.

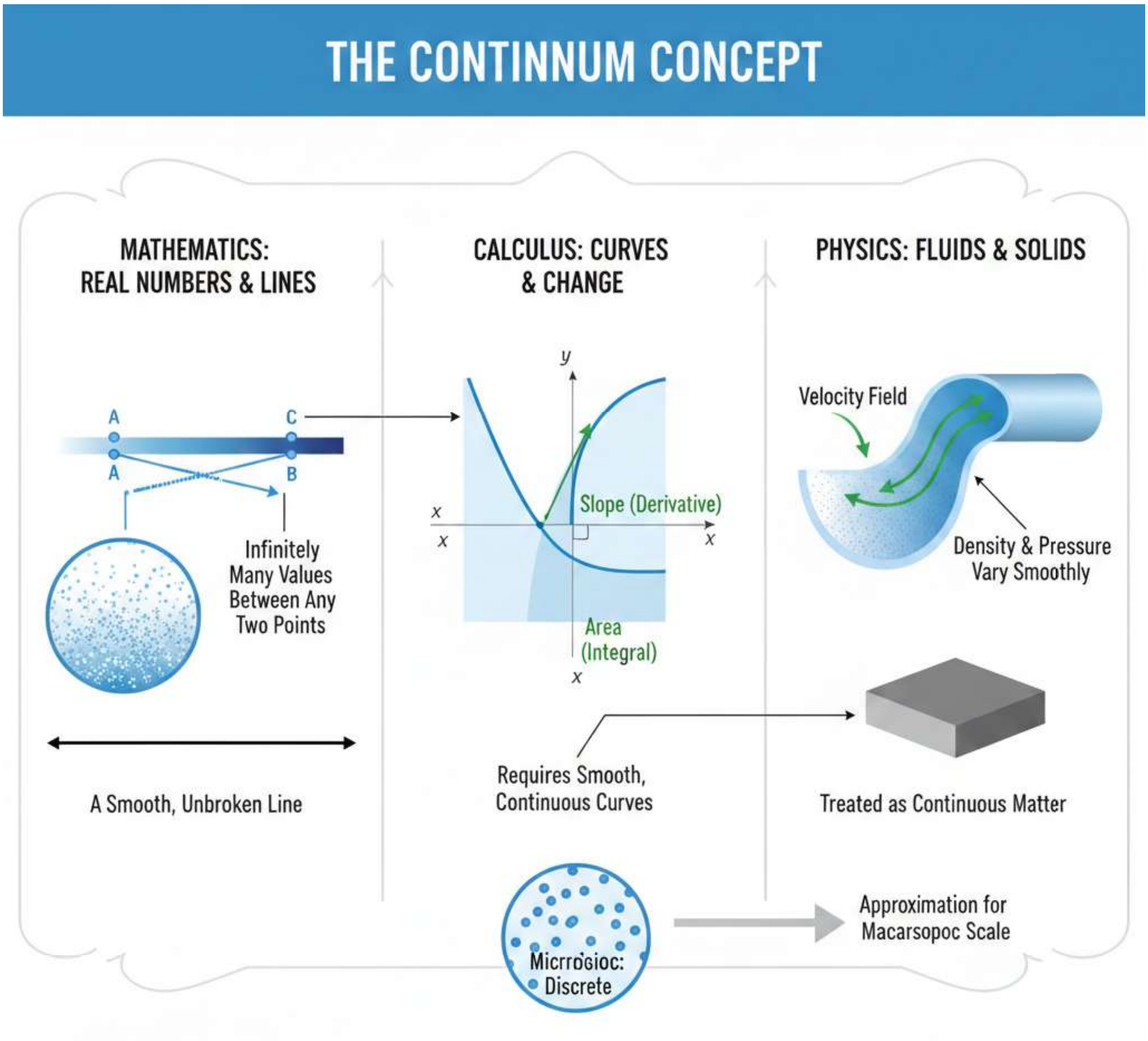


Figure 1.1: The Continuum Concept

For example, when we draw a straight line, we think of it not as a collection of separate dots but as a smooth, unbroken line made up of infinitely many points placed continuously next to each other. Similarly, in mathematics, the real number system follows the continuum concept because between any two real numbers, no matter how close they are, there exist infinitely many more numbers. This idea is extremely important in calculus, where we study limits, derivatives, and integrals. When we find the slope of a curve at a point or the area under a curve, we assume that the curve is smooth and continuous, which is only possible because of the continuum concept. In physics, this concept is used as an approximation to simplify real-world problems. Although matter is actually made up of atoms and molecules, we often treat solids and fluids as continuous materials. For instance, when studying the flow of a fluid, we assume that properties like density, pressure, and velocity vary smoothly from one point to another, rather than jumping abruptly. This assumption allows us to apply mathematical equations to describe motion, deformation, and forces. Even though at very small scales nature may behave in a discrete way, the continuum concept works very well at everyday scales and helps us understand and predict physical behavior accurately.

1.2 Homogeneity

A material is said to be homogeneous if its properties are the same at every point in the body. This means that no matter where you look inside the material, properties such as density, elasticity, or thermal conductivity remain unchanged. For example, a perfectly pure steel rod with uniform composition throughout is considered homogeneous. In homogeneity, the material does not vary from place to place, but it may still behave differently in different directions. Thus, homogeneity is related to spatial uniformity, not direction.

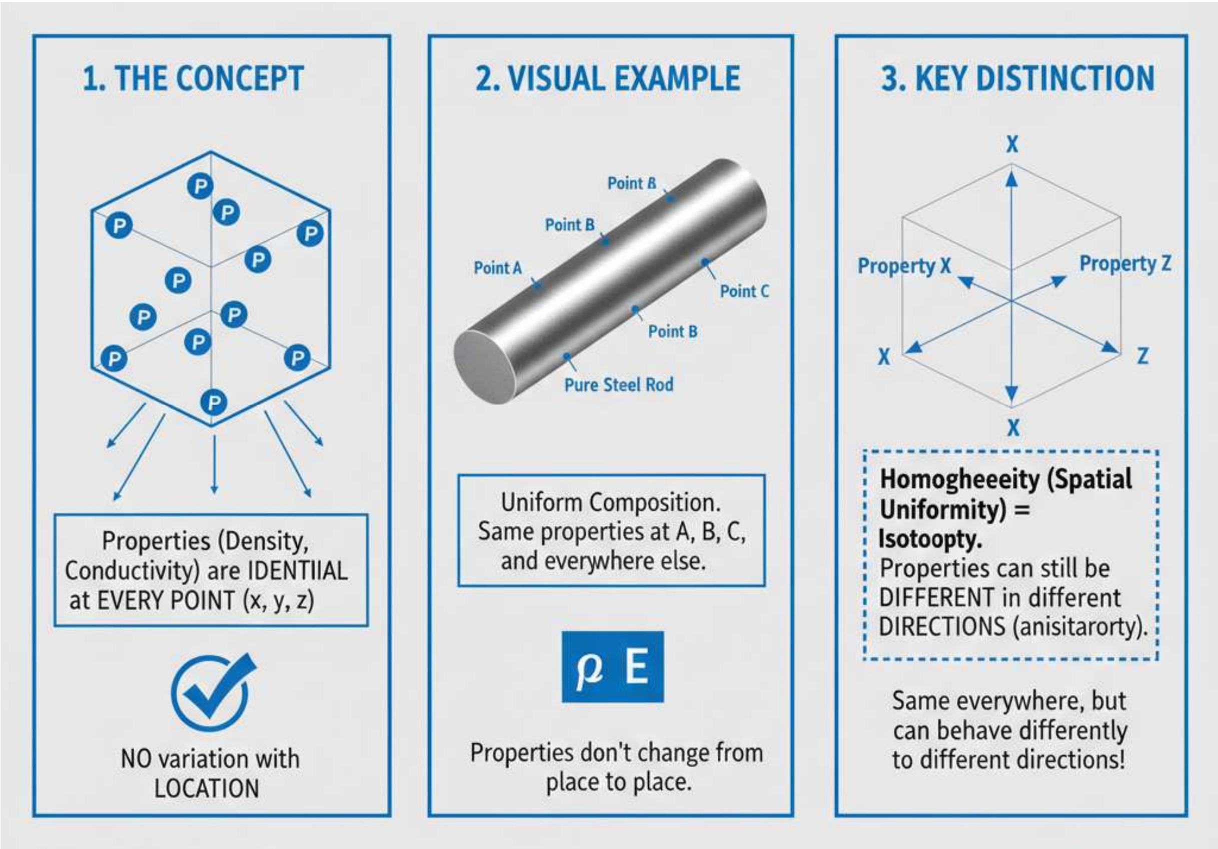


Figure 1.2: Homogeneity

1.3 Isotropy

A material is called isotropic if its properties are the same in all directions at a given point. This means that the material responds in the same way whether a force is applied along the x-direction, y-direction, or any other direction. For instance, glass and many metals (when free from defects) are often treated as isotropic materials. Isotropy is concerned with directional uniformity, not position.

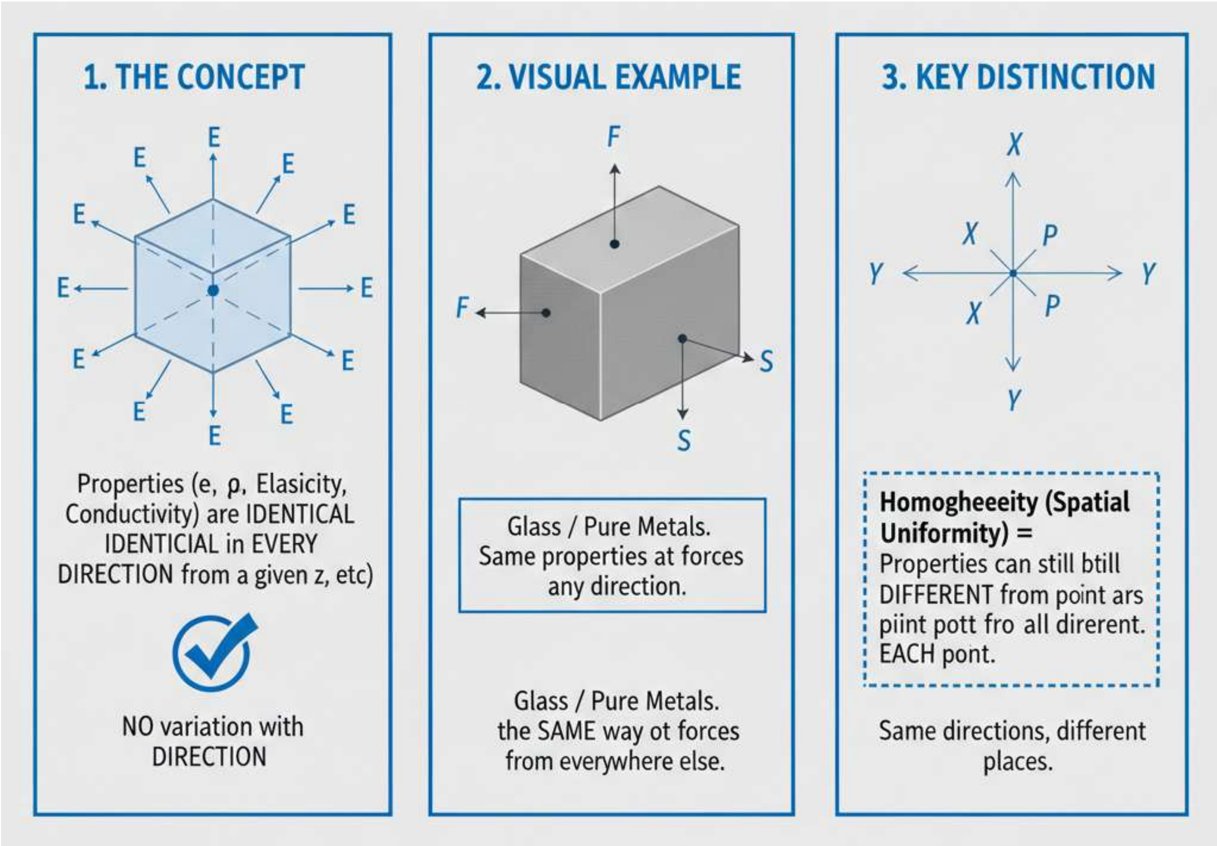


Figure 1.3: Isotropy

A material is said to be anisotropic if its properties vary with direction. In such materials, the response depends on the direction in which the load or force is applied.

1.4 Mass Density

The concept of *mass density* is developed using the *continuum concept*, which assumes that matter is continuously distributed throughout the body.

Let a body occupy a region V in space. Let us choose a small but finite volume element ΔV inside the body, containing a point P. Let the mass contained within this volume be ΔM.

The *average mass density* of the material over the volume ΔV is defined as the ratio of mass to volume

$$\rho_{\text{av}} = \frac{\Delta M}{\Delta V}$$

This definition gives only an average value and does not describe the density at a specific point. To define the density at a particular point P, we let the volume element ΔV shrink continuously around P while always containing that point. According to the continuum concept, this limiting process is meaningful.

The *mass density* at the point P is defined as the limit of the average density as the volume approaches zero:

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} \tag{2.2}$$

In the limit, the ratio becomes a derivative, and the mass density may be written as

$$\rho = \frac{dM}{dV}$$

This represents the mass per unit volume at a point. Mass density ρ is a *scalar quantity*, since it has magnitude only and no direction. In general, ρ may vary from point to point and may be expressed as a function of position:

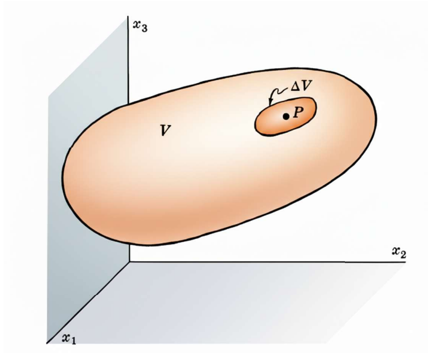


Figure 1.4: Mass-Density

$$\rho = \rho(x_1, x_2, x_3)$$

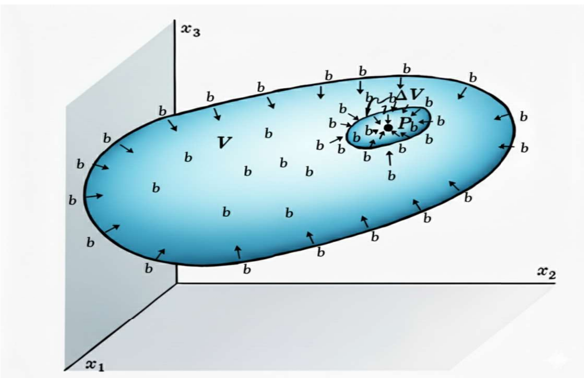
In a homogeneous material, ρ is constant throughout the body.

1.5 Forces

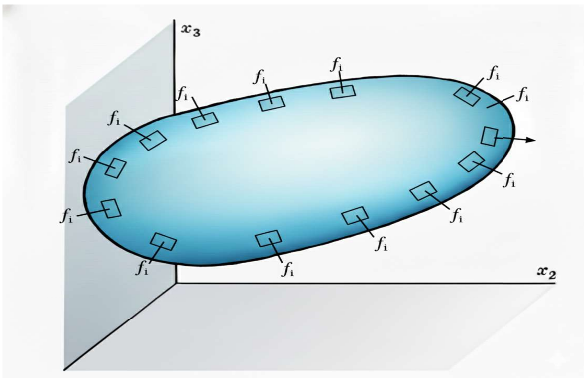
Forces acting on a continuum body are - (i) External Forces and (ii) Internal Forces.

External forces again may be of two types - (a) Body Forces (b) Surface Forces

Body Forces: *Body forces* are those forces which act on all the volume elements of a continuum. These forces are usually represented by the symbol b_i (force per unit mass) or p_i (force per unit volume). They are related through the density by the expression $p_i = \rho b_i$. E.g: Gravitational Force, Electromagnetic force fields etc.



(a) Body forces



(b) Surface forces

Surface Forces: Those forces which act on a surface element, whether it is a portion of the bounding surface of the continuum or perhaps an arbitrary internal surface, are known as *surface forces*. These are designated by f_i (force per unit area). Contact forces between bodies are a type of surface forces. E.g: Pressure Force, Friction Force, Force due to shear stress etc.

Internal forces arise as a result of the mutual interaction of pairs of particles located in the interior of the medium. According to Newton's third law, the action of one particle on another is equal in magnitude and opposite in direction to the reaction exerted by the second particle on the first. Hence, the resultant internal force is zero.

Thus, the resultant force acting on a body is equal to the sum of the total body force and the total surface force. The resultant moment about a fixed point is equal to the sum of the moments of the body forces and the surface forces about that point.

1.6 Stress Vector

Let a body occupy a volume V and be bounded by a surface S. Let a force $\Delta\vec{F}$ acts on a small surface element of area ΔS. The *stress vector* or *traction vector* $t^{\hat{n}}$ at a point on the surface is defined as the limiting value of the force per unit area, namely,

$$t^{\hat{n}} = \lim_{\Delta S \to 0} \frac{\Delta\vec{F}}{\Delta S},$$

where $\hat{n}$ denotes the unit normal to the surface element.

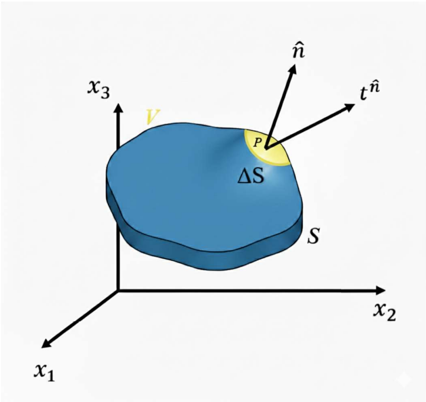


Figure 1.6: Stress Vector

- An *infinite* number of traction vectors act at a point, each acting on different surfaces through the point, defined by different normals.
- Traction vectors acting on opposite sides of a surface are equal and opposite. This can be expressed in vector form:

$$\vec{t}^{\hat{n}} = -\vec{t}^{(-\hat{n})}.$$

1.7 Cauchy Stress Principle

At any point inside a deformable body, the traction vector acting on a plane depends only on the orientation of the plane (i.e., its normal vector) and the position of the point, and not on the size or shape of the plane.
Mathematically,

$$\vec{t}^{(\hat{n})} = \boldsymbol{\sigma} \cdot \hat{n}$$

where,

$$\vec{t}^{(\hat{n})} = \text{traction vector acting on a surface with unit normal } \hat{n},$$
$$\boldsymbol{\sigma} = \text{Cauchy stress tensor (second-order tensor)}$$

In Cartesian coordinates,

$$t_i = \sigma_{ij} n_j$$

where,

- σ_{ij} are the components of the stress tensor
- n_j are the components of the unit normal vector
- t_i are the components of the traction vector

1.8 Stress Tensor

In continuum mechanics, the stress tensor is a mathematical object that describes the internal forces acting within a continuous material body.

When a body is subjected to external forces, internal forces develop between neighboring particles. These internal forces are distributed over imaginary internal surfaces inside the material. The stress tensor provides a complete description of these internal force distributions at a point.

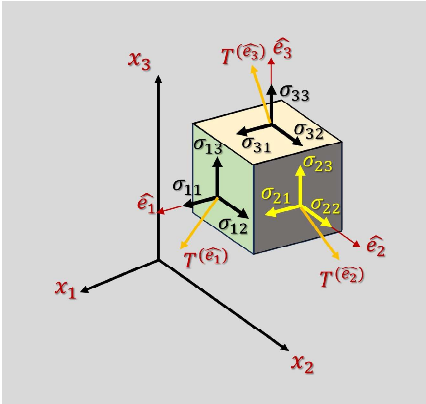


Figure 1.7: Stress Tensor

Stress Tensor in Cartesian Coordinates

The stress tensor $\boldsymbol{\sigma_{ij}}$ is a second-order tensor, which in Cartesian coordinates can be written as

$$\boldsymbol{\sigma_{ij}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Physical Meaning of Components

Each component is denoted by σ_{ij}, where:

- First index $i \rightarrow$ direction of force component
- Second index $j \rightarrow$ normal direction of the surface

Examples:

- σ_{11} : Normal stress in x-direction on a plane normal to the x-axis.
- σ_{12} : Shear stress in x-direction on a plane normal to the y-axis.
- σ_{23} : Shear stress in y-direction on a plane normal to the z-axis.

Diagonal terms $\rightarrow$ Normal stresses
Off-diagonal terms $\rightarrow$ Shear stresses

Mathematical Form

Using index notation:

$$t_i = \sigma_{ij} n_j$$

where summation over j is implied (Einstein summation convention).

This shows that the traction vector components are linear combinations of the stress tensor components and the surface normal components. Hence, the stress tensor completely determines the internal force state at a point.

Symmetry of Stress Tensor

From conservation of angular momentum (moment equilibrium), we obtain:

$$\sigma_{ij} = \sigma_{ji}$$

Thus, the stress tensor becomes symmetric. Therefore, instead of 9 independent components, the symmetric stress tensor has only 6 independent components.

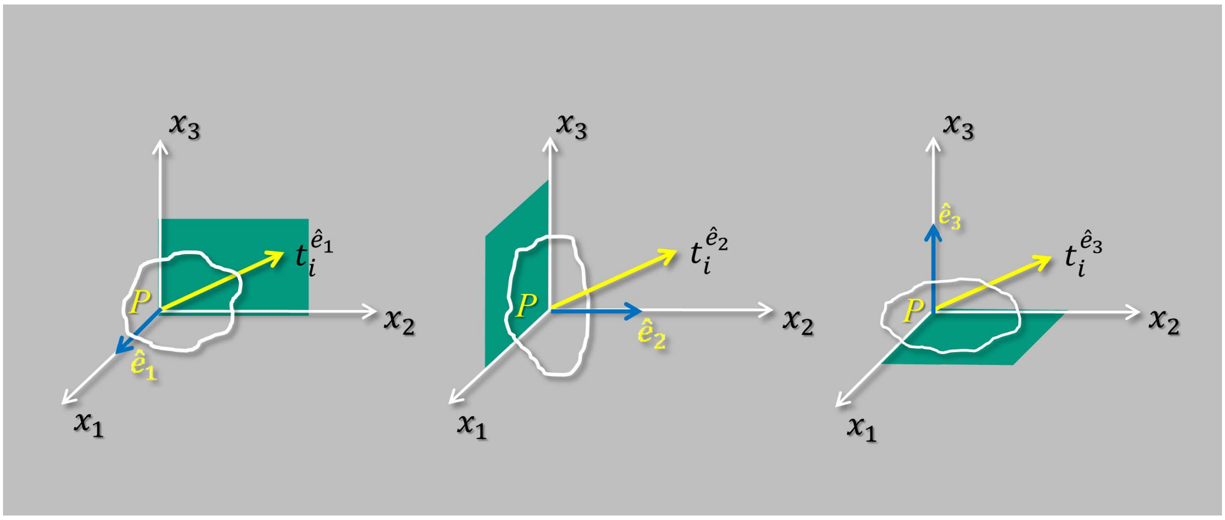
1.9 State of Stress at a Point

The collection of all possible traction vectors $\vec{t}^{(\hat{n})}$ corresponding to all possible directions $\hat{n}$ defines the **state of stress** at point P.

However, it is not necessary to know the traction vector for every possible direction. It is sufficient to know the traction vectors acting on three mutually perpendicular planes passing through P.

For convenience, we choose three planes perpendicular to the coordinate axes:

- Plane normal to $\hat{e}_1$ (the x_1-axis)
- Plane normal to $\hat{e}_2$ (the x_2-axis)
- Plane normal to $\hat{e}_3$ (the x_3-axis)



Let the corresponding traction vectors be:

$$\vec{t}^{(\hat{e}_1)}, \quad \vec{t}^{(\hat{e}_2)}, \quad \vec{t}^{(\hat{e}_3)}$$

Each traction vector can be written in terms of its Cartesian components as:

$$\vec{t}^{(\hat{e}_1)} = t_1^{(\hat{e}_1)}\hat{e}_1 + t_2^{(\hat{e}_1)}\hat{e}_2 + t_3^{(\hat{e}_1)}\hat{e}_3$$

$$\vec{t}^{(\hat{e}_2)} = t_1^{(\hat{e}_2)}\hat{e}_1 + t_2^{(\hat{e}_2)}\hat{e}_2 + t_3^{(\hat{e}_2)}\hat{e}_3$$

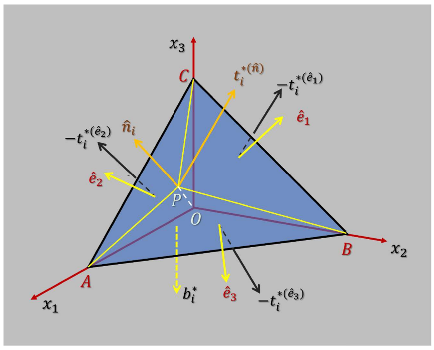
$$\vec{t}^{(\hat{e}_3)} = t_1^{(\hat{e}_3)}\hat{e}_1 + t_2^{(\hat{e}_3)}\hat{e}_2 + t_3^{(\hat{e}_3)}\hat{e}_3$$

The nine quantities $t_i^{(\hat{e}_j)}$ represent the components of stress at point P. These components form the stress tensor $\sigma_{ij} = t_i^{(\mathbf{e}_j)}$. Thus, knowing the traction vectors on three perpendicular planes is enough to completely describe the state of stress at a point.

1.10 Relation between Stress Tensor and Stress Vector

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $t_i^{(\hat{n})}$ on a plane of arbitrary orientation at that point may be established through the force equilibrium or momentum balance of a small tetrahedron PABC of the continuum, having its vertex at P.

The base of the tetrahedron PABC is taken perpendicular to $\hat{n}_i$ and the three faces are taken perpendicular to the coordinate planes. The line OP is perpendicular to the base ABC.



The components of the unit normal $\hat{n}_i$ are the direction cosine as

$$\hat{n}_1 = \cos\angle AOP$$
$$\hat{n}_2 = \cos\angle BOP$$
$$\hat{n}_3 = \cos\angle COP$$

Let, $OP = h$, then $h = OA.n_1 = OB.n_2 = OC.n_3$.

Let the area of $\triangle ABC$, $\triangle OBC$, $\triangle OCA$, $\triangle OAB$ be ds, ds_1, ds_2, ds_3 respectively.

Then the volume of the tetrahedron $\triangle V$, can be obtained by

$$\triangle V = \frac{1}{3}h.ds = \frac{1}{3}OA.ds_1 = \frac{1}{3}OB.ds_2 = \frac{1}{3}OC.ds_3$$

From this we get,

$$dS_1 = dS\frac{h}{OA} = dS\,n_1$$

$$dS_2 = dS\frac{h}{OB} = dS\,n_2$$

$$dS_3 = dS\frac{h}{OC} = dS\,n_3$$

Now, as shown in figure, the forces acting on the tetrahedron are the average stress vectors $-t_i^{*(\hat{e}_j)}$ on the faces and $t_i^{*(\hat{n})}$ on the base together with the body force b_i^* (including inertia forces, if present).

By equilibrium of forces we have,

$$t_i^{*(\hat{n})}\,dS - t_i^{*(\hat{e}_1)}\,dS_1 - t_i^{*(\hat{e}_2)}\,dS_2 - t_i^{*(\hat{e}_3)}\,dS_3 + \rho\, b_i^*\,dV = 0$$

If now the linear dimensions of the tetrahedron are reduced in a constant ratio to one another, the body forces being of higher order in the small dimensions, tend to zero more rapidly than the surface forces. At the same time, the average stress vectors approach the specific values appropriate to the designated directions at P. Therefore, by this limiting process we get,

$$t_i^{(\hat{n})}\,dS = t_i^{(\hat{e}_1)}n_1\,dS + t_i^{(\hat{e}_2)}n_2\,dS + t_i^{(\hat{e}_3)}n_3\,dS$$
$$\Rightarrow t_i^{(\hat{n})}\,dS = t_i^{(\hat{e}_j)}n_j\,dS$$
$$\Rightarrow t_i^{(\hat{n})} = t_i^{(\hat{e}_j)}n_j$$
$$\Rightarrow t_i^{(\hat{n})} = \sigma_{ji}n_j \qquad \left[t_i^{(\hat{e}_j)} = \sigma_{ji}\right]$$

In matrix notation the relationship can be written explicitly as

$$\begin{bmatrix} t_1^{(\hat{n})} & t_2^{(\hat{n})} & t_3^{(\hat{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$