Partial Differential Equations of the Second Order

Presented by

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PDE of the Second Order

Definition

A partial differential equation is said to be of the ***second order*** when it involves, ***at least one*** of the partial derivative of second order i.e., $r(=\frac{\partial^2 z}{\partial x^2}), s(=\frac{\partial^2 z}{\partial x \partial y})$ and $t(=\frac{\partial^2 z}{\partial y^2})$ but no partial derivative of third or higher order. The first order partial derivatives p and q may also be present in these equations. The second order partial differential equation is given by

$$f(x, y, z, p, q, r, s, t) = 0$$

PDE of the Second Order

The *general* linear partial differential equation of second order in two independent variables x and y, with z as dependent variable and variable coefficients is given by

$$Rr + Ss + Tt + Pp + Qq + Zz = F$$

where R, S, T, P, Q, Z and F are functions of x and y only and not all R, S, T are zero together.

Remark:

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}.$$

Origin of Second Order Partial Differential Equation by the Elimination of Arbitrary Constants

Let us consider an equation

$$f(x, y, z, a, b, c) = 0 \tag{1}$$

where z is a function of two independent variables x, y and a, b, c arbitrary constants.

Differentiating (1) partially w.r.t x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0 \tag{2}$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0 \tag{3}$$

It is not necessary that a, b and c may be eliminated between (1), (2) and (3) to give a partial differential equation. In that case differentiate (2) partially w.r.t x or differentiate (3) partially w.r.t y or differentiate either (2) partially w.r.t y or (3) partially w.r.t x. Thus we get the following equations.

Differential (2) partially w.r.t x,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial z}p + \frac{\partial f}{\partial z}\frac{\partial p}{\partial x} = 0 \tag{4}$$

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Differential (3) partially w.r.t y,

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y \partial z} q + \frac{\partial f}{\partial z} \frac{\partial q}{\partial y} = 0 \tag{5}$$

Differential (2) partially w.r.t 'y'

$$\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y \partial z} p + \frac{\partial f}{\partial z} \cdot \frac{\partial p}{\partial y} = 0 \tag{6}$$

Now eliminating a, b and c between (1), (2), (3) and one or more (if needed) of (4), (5), and (6), we shall get a second order partial differential equation.

Ex. 1. *Find a partial differential equation by eleiminating a, b, c from*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Sol: Given

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

Differentiating (1) partially w.r.t x and y successively we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad c^2x+a^2zp=0 \tag{2}$$

and

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2y + b^2zq = 0 \tag{3}$$

Here it is not possible to eliminate a, b, c between (1), (2) and (3).
So again differentiating (2) partially w.r.t x, we get

$$\begin{aligned} & c^2 + a^2\left(\frac{\partial z}{\partial x}p + z\frac{\partial p}{\partial x}\right) = 0 \\ & \text{or,} \quad c^2 + a^2(p^2 + zr) = 0 \end{aligned} \quad \Bigg| \quad \because \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = r \tag{4}$$

Now from (2)

$$c^2 = -\frac{a^2z}{x}p.$$

Substituting in (4), we get

$$-\frac{a^2 z}{x}p + a^2(p^2 + zr) = 0 \quad \text{or} \quad xzr + xp^2 - zp = 0 \tag{5}$$

Again differentiating (3) partially w.r.t y, we get

$$\begin{aligned} c^2 + b^2\left(\frac{\partial z}{\partial y}q + z\frac{\partial q}{\partial y}\right) &= 0 \\ \text{or,}\ \ c^2 + b^2(q^2 + zt) &= 0 \end{aligned} \quad \Bigg| \quad \because \ \frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = t \tag{6}$$

Now from (3),

$$c^2 = -\frac{b^2 z}{y}q.$$

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Substituting in (6), we get

$$-\frac{b^2 z}{y}q + b^2(q^2 + zt) = 0 \quad \text{or} \quad yzt + yq^2 - zq = 0 \tag{7}$$

Again differentiating (2) partially w.r.t y or (3) w.r.t x, we get

$$\begin{aligned} & a^2\left(\frac{\partial z}{\partial y}p + z\frac{\partial p}{\partial y}\right) = 0 \\ & \text{or,} \quad pq + zs = 0 \end{aligned} \quad \Bigg| \quad \because \ \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = s \tag{8}$$

The equations (5), (7) and (8) are the required second order partial differential equations.

Ex. 2. *Obtain the partial differential equation by eliminating arbitrary constants from*

$$z = Ae^{-lt}\cos mx\,\sin ny \quad \text{where } l^2 = m^2 + n^2.$$

Sol: Given,

$$z = Ae^{-lt}\cos mx\,\sin ny \tag{1}$$

Differentiating (1) partially w.r.t the independent variables t, x and y, we get

$$\frac{\partial z}{\partial t} = -Ale^{-lt}\cos mx\,\sin ny \tag{2}$$

$$\frac{\partial z}{\partial x} = -Ame^{-lt}\sin mx\,\sin ny \tag{3}$$

and

$$\frac{\partial z}{\partial y} = Ane^{-lt}\cos mx\,\cos ny \tag{4}$$

Differentiating (2), (3) and (4) partially w.r.t t, x and y respectively, we get

$$\frac{\partial^2 z}{\partial t^2} = Al^2e^{-lt}\cos mx\,\sin ny \tag{5}$$

$$\frac{\partial^2 z}{\partial x^2} = -Am^2e^{-lt}\cos mx\,\sin ny \tag{6}$$

and

$$\frac{\partial^2 z}{\partial y^2} = -An^2e^{-lt}\cos mx\,\sin ny \tag{7}$$

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$$\therefore (5) + (6) + (7) \Rightarrow \frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = A\big(l^2 - m^2 - n^2\big)e^{-lt} \cos mx \, \sin ny$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad (\because l^2 = m^2 + n^2)$$

which is the required second order partial differential equation.

Origin of Second Order Partial Differential Equation by the Elimination of Arbitrary Functions

If the given relation between the dependent variable z and two independent variables x and y contains two arbitrary functions then the elimination of these two functions from the given relation will give rise to a second order partial differential equation in general.

Ex. 1. *Form a partial differential equation by eliminating the arbitrary functions f and ϕ from*

$$z = yf(x) + x\phi(y)$$

Sol: Given,

$$z = yf(x) + x\phi(y) \tag{1}$$

Origin of Second Order PDE

Differentiating (1) partially w.r.t x and y, we get

$$\frac{\partial z}{\partial x} = yf'(x) + \phi(y) \tag{2}$$

and

$$\frac{\partial z}{\partial y} = f(x) + x\phi'(y) \tag{3}$$

Again differentiating (2) partially w.r.t y, we get

$$\frac{\partial^2 z}{\partial y\, \partial x} = f'(x) + \phi'(y) \tag{4}$$

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From (2) and (3),

$$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - \phi(y)\right] \quad \text{and} \quad \phi'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

Substituting in (4), we get

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - \phi(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

$$\Rightarrow \quad xy\frac{\partial^2 z}{\partial y\,\partial x} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \big(x\phi(y) + yf(x)\big)$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - \phi(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

$$\Rightarrow \quad xy\frac{\partial^2 z}{\partial y\,\partial x} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \big(x\phi(y) + yf(x)\big)$$

Substituting from (1),

$$xy\frac{\partial^2 z}{\partial y\,\partial x} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

$$\Rightarrow \quad xys = xp + yq - z$$

which is the required partial differential equation of order 2.

Ex. 2. *Form a partial differential equation by eliminating arbitrary functions f and g from*

$$z = f(x^2 - y) + g(x^2 + y).$$

Solution: Given,

$$z = f(x^2 - y) + g(x^2 + y) \tag{1}$$

Differentiating (1) partially w.r.t x and y, we get

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2xg'(x^2 + y) \tag{2}$$

$$\frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y) \tag{3}$$

Differentiating (2) partially w.r.t x and (3) w.r.t y, we get

$$\frac{\partial^2 z}{\partial x^2} = 4x^2 f''(x^2 - y) + 2f'(x^2 - y) + 4x^2 g''(x^2 + y) + 2g'(x^2 + y) \tag{4}$$

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$$\frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x^2 + y) \tag{5}$$

From (2),

$$f'(x^2 - y) + g'(x^2 + y) = \frac{1}{2x}\frac{\partial z}{\partial x} \tag{6}$$

Substituting from (5) and (6) in (4), we get

$$\frac{\partial^2 z}{\partial x^2} = 4x^2\frac{\partial^2 z}{\partial y^2} + \frac{1}{x}\frac{\partial z}{\partial x}$$

or

$$x\frac{\partial^2 z}{\partial x^2} = 4x^3\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} \Rightarrow xr = 4x^3 t + p$$

which is the required differential equation of order 2.

Origin of Second Order Partial Differential Equation of Special Type

Let

$$z = f(u) + g(v) + w \tag{1}$$

where f and g are functions of u and v respectively and u, v and w are prescribed functions of x and y.

Differentiating both sides of (1) partially w.r.t. x and y respectively, we get

$$p = f'(u)\, u_x + g'(v)\, v_x + w_x \tag{2}$$

and

$$q = f'(u)\, u_y + g'(v)\, v_y + w_y \tag{3}$$

Again differentiating (2) partially w.r.t x and y and differentiating (3) partially w.r.t y, we get

$$r = \frac{\partial p}{\partial x} = f''(u)\, u_x^2 + g''(v)\, v_x^2 + f'(u)\, u_{xx} + g'(v)\, v_{xx} + w_{xx} \tag{4}$$

$$s = \frac{\partial p}{\partial y} = f''(u)\, u_y u_x + g''(v)\, v_y v_x + f'(u)\, u_{xy} + g'(v)\, v_{xy} + w_{xy} \tag{5}$$

and

$$t = \frac{\partial q}{\partial y} = f''(u)\, u_y^2 + g''(v)\, v_y^2 + f'(u)\, u_{yy} + g'(v)\, v_{yy} + w_{yy} \tag{6}$$

Equations (2)–(6) can be written as

$$(p - w_x) - f'(u)u_x - g'(v)v_x = 0 \tag{7}$$

Origin of Second Order PDE

$$(q - w_y) - f'(u)u_y - g'(v)v_y = 0 \tag{8}$$

$$(r - w_{xx}) - f'(u)u_{xx} - g'(v)v_{xx} - f''(u)u_x^2 - g''(v)v_x^2 = 0 \tag{9}$$

$$(s - w_{xy}) - f'(u)u_{xy} - g'(v)v_{xy} - f''(u)u_y u_x - g''(v)v_y v_x = 0 \tag{10}$$

$$(t - w_{yy}) - f'(u)u_{yy} - g'(v)v_{yy} - f''(u)u_y^2 - g''(v)v_y^2 = 0 \tag{11}$$

Eliminating $f'(u)$, $g'(v)$, $f''(u)$ and $g''(v)$ from equation (7)–(11), we get

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t - w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0$$

Expanding with respect to the first column, we get a differential equation of the form

$$Rr + Ss + Tt + Pp + Qq = W \tag{12}$$

where R, S, T, P, Q and W are known functions of x and y.

Equation (12) is a **linear partial differential equation of second order**, and is obtained by eliminating the functions f and g. This equation (12) is of special type in which the dependent variable z does not occur.

The relation (1) is a solution of the second order linear partial differential equation (12).

Type I: Under this type, we consider equations of the form

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = f_1(x, y), \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = f_2(x, y), \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{F}{S} = f_3(x, y)$$

Equations of this type are solved by direct integration of both sides partially w.r.t. x or y as possible. When integration partially w.r.t. x is done then y is treated as constant and the constant of integration is taken as some function of y and when integration partially w.r.t. y is done then x is treated as constant and the constant of integration is taken as some function of x.

Ex. 1. *Solve* $xys = 1$.

Sol. Given,

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy}$$

Integrating both sides partially w.r.t. x, treating y constant, we get

$$\frac{\partial z}{\partial y} = \frac{1}{y}\log x + f(y)$$

Again integrating both sides partially w.r.t. y, treating x constant, we get

$$z = \log y \cdot \log x + \int f(y)\,dy + \psi(x)$$

or

$$z = \log y \cdot \log x + \phi(y) + \psi(x)$$

where $\phi(y)$ and $\psi(x)$ are arbitrary functions.

Ex. 2. Solve $xy^2 s = 1 - 2x^2 y$.

Sol. The given equation can be written as

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{1 - 2x^2 y}{xy^2} = \frac{1}{xy^2} - 2\frac{x}{y}$$

Integrating partially w.r.t. 'x', (treating y constant), we get

$$\frac{\partial z}{\partial y} = \frac{1}{y^2}\log x - \frac{x^2}{y} + f(y)$$

Again integrating partially w.r.t. y, we get

$$z = -\frac{1}{y}\log x - x^2 \log y + \int f(y)\,dy + \psi(x)$$

or

$$z = -\frac{1}{y}\log x - x^2 \log y + \phi(y) + \psi(x)$$

where $\phi(y)$ and $\psi(x)$ are arbitrary functions.

Type II: Under this type we consider equations of the following forms

$$Rr + Pp = F \quad \text{i.e.} \quad \frac{\partial p}{\partial x} + \frac{P}{R}p = \frac{F}{R}, \quad \text{Here } S = 0,\ T = Q = Z,\ R \neq 0,\ P \neq 0$$

$$Ss + Pp = F \quad \text{i.e.} \quad \frac{\partial p}{\partial y} + \frac{P}{S}p = \frac{F}{S}, \quad \text{Here } R = 0,\ T = Q = Z,\ S \neq 0,\ P \neq 0$$

$$Ss + Qq = F \quad \text{i.e.} \quad \frac{\partial q}{\partial x} + \frac{Q}{S}q = \frac{F}{S}, \quad \text{Here } R = 0,\ T = P = Z,\ S \neq 0,\ Q \neq 0$$

$$Tt + Qq = F \quad \text{i.e.} \quad \frac{\partial q}{\partial y} + \frac{Q}{T}q = \frac{F}{T}, \quad \text{Here } R = 0,\ S = P = Z,\ T \neq 0,\ Q \neq 0$$

These equations are linear equations of the form

$$\frac{dy}{dx} + Py = Q$$

whose I.F. $= e^{\int P\,dx}$
and the solution is

$$y\,e^{\int Pdx} = \int Q\,e^{\int Pdx}\,dx + C.$$

Ex. 1. *Solve* $xr + p = 9x^2y^3$.
Sol. The given equation can be written as

$$\frac{\partial p}{\partial x} + \frac{1}{x}p = 9xy^3$$

which is linear in p, whose I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$.

$$p \cdot x = \int (9y^3) \cdot x\, dx + f(y) = 3x^2y^3 + \phi(y) \quad \text{(Treating } y \text{ constant)}$$

$$\Rightarrow \quad p = \frac{\partial z}{\partial x} = 3x^2y^3 + \frac{1}{x}\phi(y)$$

Integrating partially w.r.t. x, we get

$$z = x^3y^3 + \phi(y)\log x + \psi(y)$$

where ϕ and ψ are arbitrary functions of y.
Alt: The given equation can also be written as

$$\frac{\partial}{\partial x}(xp + 1) = 9x^2y^3$$

Integrating partially w.r.t. x (treating y constant) we get

$$xp + 1 = 3x^3y^3 + f(y)$$

$$\Rightarrow \quad p = \frac{\partial z}{\partial x} = 3x^2y^3 + \frac{1}{x}\{f(y) - 1\}$$

Again integrating partially w.r.t. x (treating y constant), we get

$$z = x^3y^3 + \{f(y) - 1\}\log x + \psi(y)$$

or

$$z = x^3y^3 + \phi(y)\log x + \psi(y), \quad \text{where } \phi(y) = f(y) - 1$$

which is the required solution.

Ex. 2. *Solve* $sx + q = 4x + 2y + 2$.

Solution: The given equation can be written as

$$\frac{\partial q}{\partial x} + \frac{1}{x} q = \frac{4x + 2y + 2}{x}$$

which is linear in q, whose I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

$$q \cdot x = \int x \cdot \frac{(4x + 2y + 2)}{x} dx + f(y)$$

$$= \int (4x + 2y + 2) dx + f(y) = 2x^2 + 2(y + 1)x + f(y)$$

$$\Rightarrow \quad q = \frac{\partial z}{\partial y} = 2x + 2(y + 1) + \frac{1}{x} f(y)$$

Integrating partially w.r.t. y, we get

$$z = 2xy + y^2 + 2y + \frac{1}{x}\phi(y) + \psi(x), \quad \text{where } \int f(y)dy = \phi(y).$$

or

$$z = 2xy + y^2 + 2y + \frac{1}{x}\phi(y) + F(x), \quad \text{where } F(x) = \psi(x).$$

or

$$z = 2x^2y + xy^2 + 2xy + \phi(y) + F(x),$$

where $\phi(y)$ and $F(x)$ are arbitrary functions.

Type III: Under this type, we consider equations of the form

$$Rr + Ss + Pp = F \quad \text{or} \quad R\left(\frac{\partial p}{\partial x}\right) + S\left(\frac{\partial p}{\partial y}\right) = F - Pp$$

and

$$Ss + Tt + Qq = F \quad \text{or} \quad S\left(\frac{\partial q}{\partial x}\right) + T\left(\frac{\partial q}{\partial y}\right) = F - Qq.$$

These are linear partial differential equations of order one with p (or q) as dependent variable and x, y as independent variables.

Ex. 1. *Solve* $xy\,r + x^2 s - y\,p = x^3 e^y$.

Sol. The given equation can be written as

$$xy\frac{\partial p}{\partial x} + x^2\frac{\partial p}{\partial y} = x^3 e^y + yp \tag{1}$$

which is Lagrange's equation in p and the Lagrange's auxiliary equations are

$$\frac{dx}{xy} = \frac{dy}{x^2} = \frac{dp}{x^3 e^y + yp}.$$

Taking 1st and 2nd fractions, we get

$$x\,dx - y\,dy = 0.$$

Integrating,

$$x^2 - y^2 = c_1. \tag{2}$$

Taking 2nd and 3rd fractions, we get

$$\frac{dp}{dy} = \frac{x^3 y^2 + yp}{x^2} = xe^y + \frac{yp}{x^2}$$

or

$$\frac{dp}{dy} - \frac{y}{y^2 + c_1} p = \sqrt{(y^2 + c_1)} e^y$$

which is linear in p, whose

$$\text{I.F.} = e^{\int -\frac{y}{y^2+c_1}dy} = e^{-\frac{1}{2}\log(y^2+c_1)} = (y^2+c_1)^{-1/2}$$

$$\therefore \quad p(y^2+c_1)^{-1/2} = \int \sqrt{(y^2+c_1)e^y}(y^2+c_1)^{-1/2}dy + c_2$$

or

$$\frac{p}{\sqrt{y^2+c_1}} = \int e^y dy + c_2 = e^y + c_2$$

or

$$\frac{p}{x} = e^y + c_2$$

From (2) and (3), the solution of (1) is

$$\frac{p}{x} = e^y + f(x^2 - y^2)$$

or

$$p = \frac{\partial z}{\partial x} = xe^y + xf(x^2 - y^2)$$

Integrating partially w.r.t. x, (treating y constant) we get

$$z = e^y \int x dx + \int xf(x^2 - y^2) dx + \psi(y)$$

or

$$z = \frac{1}{2}x^2 e^y + \frac{1}{2} \int 2xf(x^2 - y^2) dx + \psi(y)$$

or

$$z = \frac{1}{2}x^2 e^y + \frac{1}{2}\phi(x^2 - y^2) + \psi(y)$$

or

$$2z = x^2 e^y + \phi(x^2 - y^2) + F(y)$$

where $2\psi(y) = F(y)$ and ϕ and F are arbitrary functions.

Ex. 2. *Solve* $yt + xs + q = 8yx^2 + 9y^2$.

Sol. The given equation can be written as

$$x\frac{\partial q}{\partial x} + y\frac{\partial q}{\partial y} = -q + 8yx^2 + 9y^2 \tag{1}$$

which is Lagrange's equation in q and the Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dq}{-q + 8yx^2 + 9y^2}$$

Taking 1st and 2nd fractions, we have

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

Integrating, $\log y - \log x = \log \frac{y}{x} = c_1$

$$\Rightarrow \frac{y}{x} = c_1 \tag{2}$$

Taking 2nd and 3rd fractions, we get

$$\frac{dq}{dy} + \frac{1}{y}q = 8x^2 + 9y = \frac{8}{c_1^2}y^2 + 9y \quad \text{using (2)}$$

or

$$\frac{dq}{dy} + \frac{1}{y}q = \frac{8}{c_1^2}y^2 + 9y$$

which is linear in q, whose I.F. $= e^{\int \frac{1}{y}dy} = e^{\log y} = y$.

$$\therefore \quad yq = \int \Big(\frac{8}{c_1^2}y^2 + 9y\Big) y\, dy + c_2 = \Big(\frac{8}{c_1^2}\Big)\frac{y^4}{4} + 3y^3 + c_2$$

$$yq = \frac{2}{c_1^2} y^4 + 3y^3 + c_2 \tag{3}$$

From (2) and (3), solution of (1) is

$$yq - \frac{2x^2}{y^2} - 3y^3 = F(y/x)$$

or

$$q = \frac{2x^2}{y} + 3y^2 + \frac{1}{y} F(y/x)$$

Integrating w.r.t. y, (treating x constant), we get

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \int \left(\frac{2x^2}{y} + 3y^2 + \frac{1}{y} F(y/x) \right) dy + \Phi(x)$$

$$z = x^2y + y^3 + \int \frac{1}{y} F(y/x)\, dy + \Phi(x)$$

Putting $y/x = t$, $(1/x)dy = dt$, we have

$$z = x^2y + y^3 + \int \frac{1}{t} F(t)\, dt + \Psi(x)$$

where

$$\int \frac{1}{t} F(t)\, dt = \Phi(t) = \Phi(y/x)$$

which is the required solution of (1), where $\Phi(y/x)$ and $\Psi(x)$ are arbitrary functions.

Type IV : Under this type, we consider equations of the forms

$$Rr + Pp + Zz = F \quad \text{i.e.,} \quad R\frac{\partial^2 z}{\partial x^2} + P\frac{\partial z}{\partial x} + Zz = F \tag{1}$$

and

$$Tt + Qq + Zz = F \quad \text{i.e.,} \quad T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} + Zz = F \tag{2}$$

These equations are ordinary linear differential equations of order two with independent variable x in (1) and y in (2).

Ex. 1. *Solve:* $r - p - \left(\frac{1}{y}\right)(\frac{1}{y} - 1)z = x^2y - x^2y^2 + 2xy^3 - 2y^3.$

Sol. Taking $D \equiv \frac{\partial}{\partial x}$, the given differential equation can be written as

$$\left[D^2 - D - \left(\frac{1}{y}\right)(\frac{1}{y} - 1)\right]z = x^2y - x^2y^2 + 2xy^3 - 2y^3 \tag{1}$$

or

$$\left[D - \tfrac{1}{y}\right]\left[D + \left(\tfrac{1}{y} - 1\right)\right] z = x^2 y - x^2 y^2 + 2xy^3 - 2y^3 \tag{2}$$

whose $\quad C.F. = e^{x}/y\,\phi_1(y) + e^{x(1-1/y)}\phi_2(y)$

To find the P.I. of (1), let

$$z = F_1 x^2 + F_2 x + F_3 \tag{3}$$

where F_1, F_2, F_3 are functions of y or constants.

$$\therefore \quad Dz = \frac{\partial z}{\partial x} = 2F_1 x + F_2$$

and

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = 2F_1$$

Putting in (1), we get

$$2F_1 - (2F_1 x + F_2) - \left(\frac{1}{y}\right)\left(\frac{1}{y} - 1\right)(F_1 x^2 + F_2 x + F_3) = x^2 y - x^2 y^2 + 2xy^3 - 2y^3$$

Equating coefficients of like powers of x, we get

$$-\left(\frac{1}{y}\right)\left(\frac{1}{y} - 1\right) F_1 = y - y^2 \tag{4}$$

$$-2F_1 - \left(\frac{1}{y}\right)\left(\frac{1}{y} - 1\right) F_2 = 2y^3 \tag{5}$$

and

$$2F_1 - F_2 - \left(\frac{1}{y}\right)\left(\frac{1}{y} - 1\right) F_3 = -2y^3 \tag{6}$$

From (4), $\quad F_1 = -y^3, \quad \therefore$ from (5) $F_2 = 0$ and from (6) $F_3 = 0$

$$\therefore \quad \text{from (3)}, \quad \text{P.I.} = -y^3 x^2.$$

Hence the required solution of (1) is

$$z = C.F. + P.I.$$

or

$$z = e^{x}/y\,\phi_1(y) + e^{x(1-1/y)}\phi_2(y) - x^2y^3.$$

THANK YOU

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