Non-Linear Partial Differential Equation of Order One

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MATHEMATICAL EXPLORATIONS

Explore Mathematical Cosmos

Types of Integrals

Types of Integrals in First Order PDEs

Consider a first-order PDE of the form

$$F(x, y, z, p, q) = 0, \quad \text{where } p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}.$$

Definition: A *complete integral* of a PDE is a solution which contains as many arbitrary constants as the number of independent variables in the PDE. For a PDE in two independent variables (x, y), the complete integral will contain two arbitrary constants, say a and b.

Example:

$$z = ax + by + ab,$$

is a complete integral of a PDE in x and y, since it involves two arbitrary constants a and b.

Definition: A *particular integral* of a PDE is obtained by assigning specific numerical values to the arbitrary constants in the complete integral. This gives one specific solution surface from the family of solutions represented by the complete integral.

Example: From the complete integral

$$z = ax + by + ab,$$

if we set $a = 1$, $b = 2$, we get

$$z = x + 2y + 2,$$

which is a particular integral.

Types of Integrals

Definition: A *singular integral* of a PDE is a special solution that cannot be obtained by simply assigning values to the arbitrary constants in the complete integral. It is obtained by eliminating the constants from the complete integral using the conditions

$$\frac{\partial z}{\partial a} = 0, \qquad \frac{\partial z}{\partial b} = 0,$$

which corresponds geometrically to finding the envelope of the family of surfaces given by the complete integral.

Example: For the complete integral

$$z = ax + by + ab,$$

eliminating a and b gives the singular integral

$$z = xy.$$

Definition: A *general integral* of a PDE is a solution which contains an arbitrary function, rather than a finite number of arbitrary constants. This represents the most general form of the solution.
Example:

$$z = f(x + y),$$

is a general integral, where f is an arbitrary function of $x + y$.

Compatible System of First–Order Equations

Let us consider first order partial differential equations

$$f(x, y, z, p, q) = 0 \tag{1}$$

and

$$g(x, y, z, p, q) = 0. \tag{2}$$

Equations (1) and (2) are known as compatible when every solution of one is also a solution of the other.

Compatible System of First–Order Equations

To find condition for (1) and (2) to be compatible.

Let,

$$J = \text{Jacobian of f and g} = \frac{\partial(f,g)}{\partial(p,q)} \neq 0 \tag{3}$$

Then (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$$p = \phi(x,y,z) \quad \text{and} \quad q = \psi(x,y,z). \tag{4}$$

Compatible System of First–Order Equations

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (4) should be completely integrable, *i.e.*, that the equation

$$dz = p\,dx + q\,dy \quad \text{or} \quad \phi\,dx + \psi\,dy - dz = 0, \quad \text{using (4).} \tag{5}$$

should be integrable. (5) is integrable if

$$\phi\left(\frac{\partial\psi}{\partial z} - 0\right) + \psi\left(0 - \frac{\partial\phi}{\partial z}\right) + (-1)\left(\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}\right) = 0$$

which is equivalent to

$$\frac{\partial\psi}{\partial x} + \phi\frac{\partial\psi}{\partial z} = \frac{\partial\phi}{\partial y} + \psi\frac{\partial\phi}{\partial z}. \tag{6}$$

Compatible System of First–Order Equations

Substituting from equations (4) in (1) and differentiating w.r.t. x and z respectively, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial \psi}{\partial x} = 0. \tag{7}$$

and

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial q}\frac{\partial \psi}{\partial z} = 0. \tag{8}$$

From (7) and (8),

$$\frac{\partial f}{\partial x} + \phi\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\left(\frac{\partial \phi}{\partial x} + \phi\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial q}\left(\frac{\partial \psi}{\partial x} + \phi\frac{\partial \psi}{\partial z}\right) = 0. \tag{9}$$

Compatible System of First–Order Equations

Similarly (2) yields

$$\frac{\partial g}{\partial x} + \phi\frac{\partial g}{\partial z} + \frac{\partial g}{\partial p}\left(\frac{\partial \phi}{\partial x} + \phi\frac{\partial \phi}{\partial z}\right) + \frac{\partial g}{\partial q}\left(\frac{\partial \psi}{\partial x} + \phi\frac{\partial \psi}{\partial z}\right) = 0. \tag{10}$$

Solving (9) and (10),

$$\frac{\partial \psi}{\partial x} + \phi\frac{\partial \psi}{\partial z} = \frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right]. \tag{11}$$

Again, substituting from equations (4) in (1) and differentiating w.r.t. y and z and proceeding as before, we obtain

$$\frac{\partial \phi}{\partial y} + \psi\frac{\partial \phi}{\partial z} = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right]. \tag{12}$$

Compatible System of First–Order Equations

Substituting from equations (11) and (12) in (6) and replacing ϕ, ψ by p, q respectively, we obtain

$$\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)}\right] = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}\right], \tag{1}$$

or

$$[f,g] = 0, \tag{13}$$

where

$$[f,g] \equiv \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}. \tag{14}$$

A PARTICULAR CASE:

To show that first order partial differential equations $p = P(x, y)$ *and* $q = Q(x, y)$ *are compatible if and only if* $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Proof. Given

$$\frac{\partial z}{\partial x} = p = P(x, y), \qquad \frac{\partial z}{\partial y} = q = Q(x, y) \tag{1}$$

Since

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy = p\,dx + q\,dy \tag{2}$$

Compatible System of First–Order Equations

it follows that the given PDEs (1) are compatible iff

$$dz = P\,dx + Q\,dy \tag{3}$$

is integrable.
Since P and Q are functions of x, y, $Pdx + Qdy$ is exact iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Therefore, (3) is integrable iff $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Compatible System of First–Order Equations

Remark 1

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, then the system of PDEs (1) is compatible and hence they possess a common solution.

Remark 2

If $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, then the system of two given partial differential equations (1) is not compatible and hence these equations possess no solution.

Ex. 1. Show that the differential equations $p = x^2 - ay$, $q = y^2 - ax$ are compatible and find their common solution.

Sol: We know that the system of equations

$$p = P(x, y), \qquad q = Q(x, y) \tag{1}$$

is compatible iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Comparing with the given system

$$p = x^2 - ay, \quad P = x^2 - ay, \tag{2}$$

$$q = y^2 - ax, \quad Q = y^2 - ax \tag{3}$$

From (2) and (3),

$$\frac{\partial P}{\partial y} = -a, \quad \frac{\partial Q}{\partial x} = -a \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

So equations (2) are compatible.

Common Solution of (2):

Substituting the values of p and q from (2) in $dz = p\,dx + q\,dy$, we get

$$dz = (x^2 - ay)\,dx + (y^2 - ax)\,dy = x^2dx + y^2dy - a\,d(xy)$$

Integrating,

$$z = \frac{x^3 + y^3}{3} - axy + c, \quad c \text{ is an arbitrary constant} \tag{4}$$

Equation (4) is the required common solution of the given equations (2).

Ex. 2. Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible and solve them.

Sol. Let

$$f(x, y, z, p, q) = xp - yq = 0 \tag{1}$$

$$g(x, y, z, p, q) = z(xp + yq) - 2xy = 0 \tag{2}$$

$$\therefore \quad \frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial p} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp - 2y & xz \end{vmatrix} = 2xy,$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial p} \\ \dfrac{\partial g}{\partial z} & \dfrac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp + yq & xz \end{vmatrix} = -x^2 p - xyq,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy,$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + y^2q.$$

Hence,

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}.$$

Compatible System of First–Order Equations

Substituting values,

$$[f, g] = 2xy - x^2p^2 - xyqp - 2xy + xypq + y^2q^2.$$

$$= -xp(xp + yq) + yq(xp + yq) = -(xp - yq)(xp + yq).$$

Using (1), we get $[f, g] = 0$. Hence, (1) and (2) are compatible.
Solving (1) and (2) for p and q, we get

$$p = \frac{y}{z}, \qquad q = \frac{x}{z}. \tag{3}$$

Using (3) in $dz = p\,dx + q\,dy$, we have

$$dz = \left(\frac{y}{z}\right) dx + \left(\frac{x}{z}\right) dy \quad \Rightarrow \quad z\,dz = d(xy)$$

Integrating,

$$\frac{z^2}{2} = xy + \frac{c}{2} \quad \text{or} \quad z^2 = 2xy + c, \quad \text{where } c \text{ is an arbitrary constant.}$$

Ex. 3. Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution.

Sol. Let

$$f(x, y, z, p, q) = xp - yq - x = 0 \tag{1}$$

$$g(x, y, z, p, q) = x^2p + q - xz = 0. \tag{2}$$

$$\therefore \quad \frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p - 1 & x \\ 2xp - z & x^2 \end{vmatrix} = (p-1)x^2 - x(2xp - z).$$

Compatible System of First–Order Equations

Similarly,

$$\frac{\partial(f,g)}{\partial(z,p)} = x^2, \quad \frac{\partial(f,g)}{\partial(y,q)} = -q, \quad \frac{\partial(f,g)}{\partial(z,q)} = -xy.$$

$$\begin{aligned}
\therefore [f,g] &= \frac{\partial(f,g)}{\partial(x,p)}p + \frac{\partial(f,g)}{\partial(y,q)}q + \frac{\partial(f,g)}{\partial(z,p)}p + \frac{\partial(f,g)}{\partial(z,q)}q \\
&= (p-1)x^2 - x(2xp - z) - px^2 - q - xyq \\
&= -x^2 + xz - q - xyq = -x^2 + xz - qxy, \quad \text{by (2)} \\
&= x(-x+z) + (xp - yq) = 0, \quad \text{by (1)}.
\end{aligned}$$

Hence (1) and (2) are compatible.

Compatible System of First–Order Equations

Solving (1) and (2) for p and q,

$$p = \frac{1 + yz}{1 + xy}, \qquad q = \frac{x(z - x)}{1 + xy}. \tag{3}$$

Using (3) in $dz = pdx + qdy$,

$$dz = \frac{(1 + yz)}{(1 + xy)}dx + \frac{x(z - x)}{(1 + xy)}dy$$
$$\Rightarrow (1 + xy)dz = (1 + yz)dx + x(z - x)dy$$
$$\Rightarrow (1 + xy)dz - z(ydx + xdy) = dx - x^2dy$$
$$\Rightarrow \frac{(1 + xy)\,dz - z\,d(xy)}{(1 + xy)^2} = \frac{dx - x^2dy}{(1 + xy)^2} = \frac{\frac{dx}{x^2} - dy}{(y + 1/x)^2} \quad \Rightarrow d\left(\frac{z}{1 + xy}\right) = -\frac{d(y + 1/x)}{(y + 1/x)^2}$$

Compatible System of First–Order Equations

Integrating it, $$\frac{z}{1+xy} = \frac{1}{y+1/x} + c \qquad \text{or} \qquad \frac{z}{1+xy} = \frac{x}{1+xy} + c.$$

$$\therefore \quad z - x = c(1+xy), \quad c \text{ being an arbitrary constant.}$$

Charpit's Method

Charpit's Method (General method of solving partial differential equations of order one but of any degree)

Let the given partial differential equation of first order and non–linear in p and q be

$$f(x, y, z, p, q) = 0 \tag{1}$$

and

$$dz = p\,dx + q\,dy \tag{2}$$

We know that the next step consists in finding another relation

$$F(x, y, z, p, q) = 0 \tag{3}$$

such that when the values of p and q obtained by solving (1) and (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiate partially (1) and (3) with respect to x and y and get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0, \tag{4}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0, \tag{5}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0, \tag{6}$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \tag{7}$$

Eliminating $\frac{\partial p}{\partial x}$ from (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

or equivalently

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \tag{8}$$

Charpit's Method

Similarly, eliminating $\frac{\partial q}{\partial y}$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0 \qquad (9)$$

Since $\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial p}{\partial y}$, the last term in (8) is the same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9).
Therefore, adding (8) and (9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q= \qquad (10)$$

Charpit's Method

This is a linear equation of the first order to obtain the desired function F. Integral of (10) is obtained by solving the auxiliary equations

$$\frac{dp}{\left(\frac{\partial f}{\partial x}\right) + p\left(\frac{\partial f}{\partial z}\right)} = \frac{dq}{\left(\frac{\partial f}{\partial y}\right) + q\left(\frac{\partial f}{\partial z}\right)} = \frac{dz}{-p\left(\frac{\partial f}{\partial p}\right) - q\left(\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad (11)$$

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral.

Note. In what follows we shall use the following standard notations:

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial f}{\partial y} = f_y, \quad \frac{\partial f}{\partial z} = f_z, \quad \frac{\partial f}{\partial p} = f_p, \quad \frac{\partial f}{\partial q} = f_q$$

Therefore, Charpit's auxiliary equations (11) may be re–written as

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0} \qquad (11')$$

Working Rule While Using Charpit's Method

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations.

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \qquad (11')$$

Step 3. Using the value of f in step 1 write down the values of $\partial f/\partial x$, $\partial f/\partial y$, $\ldots$, *i.e.*, $f_x, f_y, \ldots$ etc. occurring in step 2 and put these in Charpit's auxiliary equations.
Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.
Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in

$$dz = p\,dx + q\,dy$$

which on integration gives the complete integral of the given equation.

The Singular and General integrals may be obtained in the usual manner.
Remark. Sometimes Charpit's equations give rise to $p = a$ and $q = b$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

Charpit's Method

Ex. 1. Find a complete integral of $z = px + qy + p^2 + q^2$.

Sol. Let

$$f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0 \tag{1}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \tag{2}$$

From (1),

$$f_x = -p, \quad f_y = -q, \quad f_z = 1, \quad f_p = -x - 2p, \quad f_q = -y - 2q \tag{3}$$

Using (3), (2) reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q}. \tag{4}$$

Taking the first fraction of (4), $dp = 0$ so that

$$p = a. \tag{5}$$

Taking the second fraction of (4), $dq = 0$ so that

$$q = b. \tag{6}$$

Putting $p = a$ and $q = b$ in (1), the required complete integral is

$$z = ax + by + a^2 + b^2,$$

where a, b are arbitrary constants.

Ex. 2. Find a complete integral of $px + qy = pq$.

Sol. Let,

$$f(x, y, z, p, q) \equiv px + qy - pq = 0 \tag{1}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p + p \cdot 0} = \frac{dq}{q + q \cdot 0}. \tag{2}$$

Taking the last two fractions of (2),

$$\frac{1}{p}dp = \frac{1}{q}dq$$

Charpit's Method

Integrating,

$$\log p = \log q + \log a \quad \text{or} \quad p = aq \tag{3}$$

Substituting this value of p in (1), we have

$$aqx + qy - aq^2 = 0 \quad \Rightarrow \quad aq = ax + y, \quad (q \neq 0). \tag{4}$$

$$\therefore \quad q = \frac{ax + y}{a}, \qquad p = ax + y \tag{5}$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (ax + y)dx + \frac{(ax + y)}{a}\,dy,$$

or,

$$adz = (ax + y)(a\,dx + dy)$$

Let, $u = ax + y$, then

$$adz = (ax + y)(a\,dx + dy) = u\,du$$

Integrating,

$$az = \frac{u^2}{2} + b$$

Hence,

$$az = \frac{(ax + y)^2}{2} + b,$$

which is a complete integral, a and b being arbitrary constants.

Ex. 3. *Find a complete, singular and general integrals of* $(p^2 + q^2)y = qz$.

Sol. Let,

$$f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0 \tag{1}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

or

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}, \quad \text{by (1)} \tag{2}$$

Taking the first two fractions, we get

$$2pdp + 2qdq = 0 \quad \Rightarrow \quad p^2 + q^2 = a \tag{3}$$

Charpit's Method

Using (3), (1) gives

$$a^2 y = qz$$

Putting this value of q in (3), we get

$$p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \left(\frac{a^2 y}{z}\right)^2} = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$$

Now putting these values of p and q in $dz = pdx + qdy$, we have

$$dz = \frac{a}{z}\sqrt{z^2 - a^2 y^2}\, dx + \frac{a^2 y}{z}\, dy,$$

or

$$\frac{zdz - a^2 y\, dy}{\sqrt{z^2 - a^2 y^2}} = a\, dx$$

Integrating,

$$(z^2 - a^2y^2)^{1/2} = ax + b \quad \text{or} \quad z^2 - a^2y^2 = (ax+b)^2, \tag{4}$$

which is a required complete integral, a, b being arbitrary constants.
Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2(ax+b)x \tag{5}$$

and

$$0 = 2(ax+b). \tag{6}$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \tag{7}$$

Differentiating (7) partially w.r.t. a,

$$-2ay^2 = 2[ax + \phi(a)] \cdot [x + \phi'(a)]. \tag{8}$$

General integral is obtained by eliminating a from (7) and (8).

Charpit's Method: Standard Form I

Standard Form I. Only p and q present.
Under this standard form, we consider equations of the form

$$f(p,q) = 0. \tag{1}$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q},$$

giving

$$\frac{dp}{0} = \frac{dq}{0}, \quad \text{by (1).}$$

Charpit's Method: Standard Form I

Taking the first ratio, $dp = 0$ so that

$$p = \text{constant} = a, \ \text{say}. \tag{2}$$

Substituting in (1), we get

$$f(a, q) = 0, \quad \Rightarrow \quad q = \text{constant} = b, \ \text{say}, \tag{3}$$

where b is such that

$$f(a, b) = 0.$$

Then,

$$dz = p\,dx + q\,dy = adx + bdy, \quad \text{using (2) and (3).}$$

Charpit's Method: Standard Form I

Integrating,

$$z = ax + by + c, \tag{5}$$

where c is an arbitrary constant. (5) together with (4) give the required solution. Now solving (4) for b, suppose we obtain

$$b = F(a), \text{ say.}$$

Putting this value of b in (5), the *complete integral* of (1) is

$$z = ax + yF(a) + c, \tag{6}$$

which contains two arbitrary constants a and c which are equal to the number of independent variables, namely x and y.

Charpit's Method: Standard Form I

The *singular integral* of (1) is obtained by eliminating a and c between the complete integral (6) and the equations obtained by differentiating (6) partially w.r.t. a and c; i.e., between

$$z = ax + yF(a) + c, \quad 0 = x + yF'(a), \quad 0 = 1. \tag{7}$$

Since the last equation in (7) is meaningless, we conclude that the equations of standard form I have no singular solution.

In order to find the *general integral* of (1), we first take $c = \phi(a)$ in (6), ϕ being an arbitrary function, and obtain

$$z = ax + yF(a) + \phi(a). \tag{8}$$

Now, we differentiate (8) partially with respect to a and get

$$0 = x + yF'(a) + \phi'(a). \tag{9}$$

Charpit's Method: Standard Form I

Eliminating a between (8) and (9), we get the general solution of (1).

Remark. Sometimes change of variables can be employed to transform a given equation to standard form I.

Ex. 1. *Solve: $p^2 + q^2 = m^2$, where m is a constant.*

Sol. Given that

$$p^2 + q^2 = m^2. \tag{1}$$

Since (1) is of the form $f(p, q) = 0$, its solution is

$$z = ax + by + c, \tag{2}$$

where $a^2 + b^2 = m^2$ or $b = (m^2 - a^2)^{1/2}$, on putting a for p and b for q in (1).

$$\therefore \text{ From (2), the complete integral is } \quad z = ax + y(m^2 - a^2)^{1/2} + c, \tag{3}$$

Charpit's Method: Standard Form I

which contains two arbitrary constants a and c.
For singular solution, differentiating (3) partially with respect to a and c, we get

$$0 = x - \frac{ay}{(m^2 - a^2)^{1/2}}, \quad \text{and} \quad 0 = 1.$$

But $0 = 1$ is absurd. Hence there is no singular solution of (1).
For general solution, let $c = \phi(a)$ in (3). Then, we get

$$z = ax + y(m^2 - a^2)^{1/2} + \phi(a). \tag{4}$$

Differentiating (4) partially with respect to a, we get

$$0 = a - \frac{ay}{(m^2 - a^2)^{1/2}} + \phi'(a). \tag{5}$$

Eliminating a from (4) and (5), we get the required general solution.

Equations Reducible to Standard Form I

Ex. 1. *Find the complete integral of* $x^2p^2 + y^2q^2 = z$

Sol. The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{x}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1. \tag{1}$$

Let,

$$(1/x)dx = dX, \quad (1/y)dy = dY, \quad (1/\sqrt{z})dz = dZ,$$

so that

$$\log x = X, \quad \log y = Y, \quad 2\sqrt{z} = Z. \tag{2}$$

Using (2), (1) becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1, \tag{4}$$

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$.
(4) is of the form $f(P, Q) = 0$.
$\therefore$ Solution of (4) is

$$Z = aX + bY + c, \tag{5}$$

where $a^2 + b^2 = 1 \quad \text{or} \quad b = \sqrt{1 - a^2}$, on putting a for P and b for Q in (4).
$\therefore$ From (5), the required complete integral is

$$Z = aX + Y\sqrt{1 - a^2} + c \quad \text{or} \quad 2\sqrt{z} = a \log x + \log y \cdot \sqrt{1 - a^2} + c. \tag{3}$$

or

$$\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z}, \quad \text{taking } c = -\log c',$$

or

$$\log\{x^a y^{\sqrt{1-a^2}}/c'\} = 2\sqrt{z},$$

Equations Reducible to Standard Form I

where a and c' are two arbitrary constants.

$$\therefore \quad x^a y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}.$$

Standard Form II. Clairaut Equation

A first order partial differential equation is said to be of *Clairaut form* if it can be written in the form

$$z = px + qy + f(p,q) \tag{1}$$

Let

$$F(x,y,z,p,q) \equiv px + qy + f(p,q) - z \tag{2}$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px - qy - p\left(\frac{\partial f}{\partial p}\right) - q\left(\frac{\partial f}{\partial q}\right)} = \frac{dx}{-x - \left(\frac{\partial f}{\partial p}\right)} = \frac{dy}{-y - \left(\frac{\partial f}{\partial q}\right)}, \quad \text{by (1)}$$

Then, first and second fractions $\implies dp = 0$ and $dq = 0 \implies p = a,\ q = b$.
Substituting these values in (1), the complete integral is

$$z = ax + by + f(a, b)$$

Remark 1. Observe that the complete integral of (1) is obtained by merely replacing p and q by a and b respectively. Singular and general integrals can be obtained by usual methods.
Remark 2. Sometimes change of variables can be employed to transform a given equation to standard form II.

Ex. 1. *Solve* $z = px + qy + pq$.
Sol. The complete integral is

$$z = ax + by + ab, \quad a, b \text{ being arbitrary constants} \tag{1}$$

Standard Form II. Clairaut Equation

Singular integral. Differentiating (1) partially w.r.t. a and b, we have

$$0 = y + a, \tag{2}$$

and

$$a = x + b.$$

Eliminating a and b between (1) and (2), we get

$$z = -xy - xy + xy \quad i.e., \quad z = -xy,$$

which is the required singular solution, for it satisfies the given equation.
General Integral. Let, $b = \phi(a)$, where ϕ denotes an arbitrary function.
Then (1) becomes

$$z = ax + \phi(a)y + a\,\phi(a). \tag{3}$$

Standard Form II. Clairaut Equation

Differentiating (3) partially w.r.t. a,

$$0 = x + \phi'(a)y + \phi(a) - a\,\phi'(a). \tag{4}$$

The general integral is obtained by eliminating a between (3) and (4).

Standard Form III. Only p, q and z present.

Under this standard form we consider differential equation of the form

$$f(p, q, z) = 0 \tag{1}$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

or

$$\frac{dp}{p\left(\frac{\partial f}{\partial z}\right)} = \frac{dq}{q\left(\frac{\partial f}{\partial z}\right)} = \frac{dz}{-p\left(\frac{\partial f}{\partial p}\right) - q\left(\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}, \quad \text{using (1)}$$

Taking the first two ratios,

$$(1/p)dp = (1/q)dq$$

Standard Form III. Only p, q and z present.

Integrating,

$$q = ap, \quad a \text{ being an arbitrary constant.} \tag{2}$$

Now,

$$dz = p\,dx + q\,dy = p\,dx + ap\,dy, \quad \text{using (2)}$$

or

$$dz = p(dx + ady) = pd(x + ay) = p\,du, \tag{3}$$

where

$$u = x + ay. \tag{4}$$

Now, (3) $\Rightarrow \quad p = \frac{dz}{du} \quad$ and so by (2) $\quad q = ap = a\left(\frac{dz}{du}\right)$.

Standard Form III. Only p, q and z present.

Substituting these values of p and q in (1), we get

$$f\left(\frac{dz}{du},\ a\frac{dz}{du},\ z\right) = 0, \tag{5}$$

which is an ordinary differential equation of first order. Solving (5), we get z as a function of u. Complete integral is then obtained by replacing u by $(x + ay)$.

Working rule for solving equations of the form $f(p, q, z) = 0$

Step I. Let,

$$f(p, q, z) = 0 \tag{1}$$

and set

$$u = x + ay, \qquad a \text{ is an arbitrary constant.} \tag{2}$$

Step II. Replace p and q by $\dfrac{dz}{du}$ and $a\dfrac{dz}{du}$ respectively in (1) and solve the resulting ordinary differential equation of first order by usual methods.
Step III. Replace u by $x + ay$ in the solution obtained in Step II.

Ex. 1. *Solve* $p^2 + q^2 = z$.
Sol. Given equation is

$$p^2 + q^2 = z. \tag{1}$$

which is of the form $f(p, q, z) = 0$.
Let $u = x + ay$, where a is an arbitrary constant.

Standard Form III. Only p, q and z present.

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (1), we have

$$\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z \quad \text{or} \quad \left(\frac{dz}{du}\right)^2 = \frac{z}{(1+a^2)}.$$

Thus,

$$\frac{dz}{du} = \pm\frac{z^{1/2}}{(1+a^2)^{1/2}} \quad \text{or} \quad \pm\, z^{-1/2}(1+a^2)^{1/2}\,dz = du.$$

Integrating,

$$\pm 2z^{1/2}(1+a^2)^{1/2} = u + b \quad \text{or} \quad \pm\, 2z^{1/2}(1+a^2)^{1/2} = x + ay + b.$$

Thus,

$$4z(1+a^2) = (x+ay+b)^2, \quad a, b \text{ being arbitrary constants.} \tag{2}$$

Standard Form III. Only p, q and z present.

(2) is the *complete integral* of the given equation (1).
Differentiating (2) partially w.r.t. a and b, we get

$$8az = 2y(x + ay + b) \quad \text{or} \quad 4az = y(x + ay + b), \tag{3}$$

$$0 = 2(x + ay + b) \quad \text{or} \quad x + ay + b = 0. \tag{4}$$

Substituting the value of $(x + ay + b)$ from (4) in (3), we have

$$4az = 0 \quad \text{or} \quad z = 0,$$

which is the *singular solution*.
In order to get the general solution, let us put $b = \psi(a)$ in (2) and get

$$4z(1 + a^2) - \{x + ay + \psi(a)\}^2 = 0. \tag{5}$$

Standard Form III. Only p, q and z present.

Differentiating (5) partially w.r.t. a,

$$8az - 2\{x + ay + \psi(a)\}\{y + \psi'(a)\} = 0. \tag{6}$$

The required *general solution* is given by (5) and (6).

In this form z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.
Let,

$$F(x,y,z,p,q) = f_1(x,p) - f_2(y,q) = 0. \tag{1}$$

Then Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or

$$\frac{dp}{\frac{\partial f_1}{\partial x}} = \frac{dq}{-\frac{\partial f_2}{\partial y}} = \frac{dz}{-p(\partial f_1/\partial p) + q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}, \quad \text{by (1).}$$

Taking the first and the fourth ratios, we have

$$\left(\frac{\partial f_1}{\partial p}\right) dp + \left(\frac{\partial f_1}{\partial x}\right) dx = 0 \quad \text{or} \quad df_1 = 0.$$

Integrating,

$$f_1 = a, \quad a \text{ being an arbitrary constant.}$$

$$\therefore \quad (1) \;\Rightarrow\; f_1(x,p) = f_2(y,q) = a. \tag{2}$$

Now,

$$(2) \;\Rightarrow\; f_1(x,p) = a \quad \text{and} \quad f_2(y,q) = a. \tag{3}$$

From (3), on solving for p and q respectively, we get

$$p = F_1(x,a), \quad q = F_2(y,a), \tag{4}$$

Standard form IV. Equation of the form $f_1(x,p) = f_2(y,q)$

Substituting these values in $dz = p\,dx + q\,dy$, we get

$$dz = F_1(x,a)\,dx + F_2(y,a)\,dy.$$

Integrating,

$$z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + b,$$

which is a complete integral containing two arbitrary constants a and b.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form IV.
Remark 2. Singular and general integral are obtained by well known methods.

Ex. 1. *Find a complete integral of* $x(1+y)p = y(1+x)q$.

Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$

Sol. Separating p and x from q and y, the given equation reduces to

$$\frac{xp}{1+x} = \frac{yq}{1+y}$$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \quad \text{and} \quad \frac{yq}{1+y} = a$$

so that

$$p = a\left(\frac{1+x}{x}\right) \quad \text{and} \quad q = a\left(\frac{1+y}{y}\right)$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}dy = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy$$

Integrating,

$$z = a(\log x + x) + a(\log y + y) + b = a(\log xy + x + y) + b,$$

which is a complete integral containing two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p^2 + q^2 = z^2(x+y)$.

Sol. Given

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2(x+y)$$

Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$

or

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2 = x + y \tag{1}$$

Let

$$(1/z)dz = dZ \quad \text{so that} \quad \log z = Z. \tag{2}$$

Using (2), (1) becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y,$$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = y - Q^2.$$

Standard form IV. Equation of the form $f_1(x,p) = f_2(y,q)$

Equating each side to an arbitrary constant a, we have

$$P^2 - x = a \quad \text{and} \quad y - Q^2 = a$$

so that

$$P = \sqrt{a+x}, \quad Q = \sqrt{y-a}.$$

Putting these values of P and Q in

$$dZ = P\,dx + Q\,dy, \quad \Rightarrow \quad dZ = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy.$$

Integrating,

$$Z = \frac{2}{3}\left[(a+x)^{3/2} + (y-a)^{3/2}\right] + \frac{2}{3}b.$$

$\therefore\ \log z = \frac{2}{3}\left[(a+x)^{3/2} + (y-a)^{3/2} + b\right]$ is a complete integral, using $Z = \log z$.

JACOBI'S METHOD

This method is used for solving partial differential equations involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables. We begin with the case of three independent variables. The results arrived at are, however, general and will be used with suitable modification for the case of four independent variables and so on.

Let

$$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}.$$

Consider a partial differential equation

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0. \tag{1}$$

JACOBI'S METHOD

The main idea in Jacobi's method is to get two additional partial differential equations of the first order

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1, \tag{2}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2, \tag{3}$$

where a_1 and a_2 are two arbitrary constants such that (1), (2) and (3) can be solved for p_1, p_2, p_3 in terms of x_1, x_2, x_3 which when substituted in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3, \tag{4}$$

makes it integrable, for which the conditions are

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial p_1}{\partial x_2}, \quad \frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3}, \quad \frac{\partial p_1}{\partial x_3} = \frac{\partial p_3}{\partial x_1}. \tag{5}$$

Differentiating (1) and (2) partially w.r.t. x_1, we have

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0, \tag{6}$$

and

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0. \tag{7}$$

Eliminating $\frac{\partial p_1}{\partial x_1}$ from (6) and (7), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial x_1}\frac{\partial f}{\partial p_1}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial p_2}\frac{\partial f}{\partial p_1}\right)\frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1} - \frac{\partial F_1}{\partial p_3}\frac{\partial f}{\partial p_1}\right)\frac{\partial p_3}{\partial x_1} = 0. \tag{8}$$

Similarly, differentiating (1) and (2) partially w.r.t. x_2 and then eliminating $\frac{\partial p_2}{\partial x_2}$ from the resulting equations, we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial F_1}{\partial x_2}\frac{\partial f}{\partial p_2}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}-\frac{\partial F_1}{\partial p_1}\frac{\partial f}{\partial p_2}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}-\frac{\partial F_1}{\partial p_3}\frac{\partial f}{\partial p_2}\right)\frac{\partial p_3}{\partial x_2}=0. \tag{9}$$

Again, differentiating (1) and (2) partially w.r.t. x_3 and then eliminating $\frac{\partial p_3}{\partial x_3}$ from the resulting equations, we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial F_1}{\partial x_3}\frac{\partial f}{\partial p_3}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}-\frac{\partial F_1}{\partial p_1}\frac{\partial f}{\partial p_3}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}-\frac{\partial F_1}{\partial p_2}\frac{\partial f}{\partial p_3}\right)\frac{\partial p_2}{\partial x_3}=0. \tag{10}$$

Adding (8), (9) and (10) and using the relations (5), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right) + \left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right) + \left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right) = 0. \quad (11)$$

The L.H.S. of (11) is generally denoted by (f, F_1). Then, (11) becomes

$$(f, F_1) = \sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right) = 0. \quad (11')$$

Starting with (1) and (3) in place of (1) and (2) and proceeding as above, we have a similar relation

$$(f, F_2) = \sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0. \quad (12)$$

JACOBI'S METHOD

Again, starting with (2) and (3) in place of (1) and (2) and proceeding as above, we again have a similar relation

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0. \tag{13}$$

(11) [or (11′)] and (12) are linear equations of first order with $x_1, x_2, x_3, p_1, p_2, p_3$ as independent variables and F_1, F_2 as dependent variables respectively. For both of these equations, Lagrange's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = -\frac{dx_1}{\partial f/\partial p_1}, \quad \frac{dp_2}{\partial f/\partial x_2} = -\frac{dx_2}{\partial f/\partial p_2}, \quad \frac{dp_3}{\partial f/\partial x_3} = -\frac{dx_3}{\partial f/\partial p_3}, \tag{14}$$

which are known as *Jacobi's auxiliary equations*.

We try to find two independent integrals

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$$

with help of (14). If these relations satisfy (13), these are the required two additional relations (2) and (3).

We now solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Substituting these values in (4) and then integrating the resulting equation, we shall obtain a complete integral of the given equation containing three arbitrary constants of integration.

Working rules for solving partial differential equations with three or more independent variables. Jacobi's method

Step I: Suppose the given equation with three independent variables is

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \tag{1}$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{-dx_1}{\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{-dx_2}{\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{-dx_3}{\partial f/\partial p_3}$$

Solving these equations we obtain two additional equations

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \tag{2}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2, \tag{3}$$

where a_1 and a_2 are arbitrary constants.
While obtaining (2) and (3), try to select simple equations so that later on solutions of (1), (2) and (3) may be as easy as possible.
Step III. Verify that relations (2) and (3) satisfy the condition

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0. \tag{4}$$

If (4) is satisfied then solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Their substitution in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

and subsequent integration leads to a complete integral of the given equation.

Remark 1. Sometime, change of variables can be employed to reduce the given equation in a form solvable by Jacobian method.

Remark 2. While solving a partial differential equation with four independent variables, we modify the above working rule as follows:

Step I. Suppose the given equation with four independent variables is

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0. \tag{1}$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{-dx_1}{\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{-dx_2}{\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{-dx_3}{\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4} = \frac{-dx_4}{\partial f/\partial p_4}.$$

Solving these equations we obtain three additional equations

$$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1 \tag{2}$$

$$F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2 \tag{3}$$

and

$$F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3, \tag{4}$$

where a_1, a_2 and a_3 are arbitrary constants.

Step III. Verify that relations (2), (3) and (4) satisfy following three conditions:

$$(F_1, F_2) = \sum_{r=1}^{4} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0, \tag{5}$$

$$(F_2, F_3) = \sum_{r=1}^{4} \left(\frac{\partial F_2}{\partial x_r} \frac{\partial F_3}{\partial p_r} - \frac{\partial F_2}{\partial p_r} \frac{\partial F_3}{\partial x_r} \right) = 0, \tag{6}$$

$$(F_3, F_1) = \sum_{r=1}^{4} \left(\frac{\partial F_3}{\partial x_r} \frac{\partial F_1}{\partial p_r} - \frac{\partial F_3}{\partial p_r} \frac{\partial F_1}{\partial x_r} \right) = 0. \tag{7}$$

If (5), (6) and (7) are satisfied, then solve (1), (2), (3) and (4) for p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4. Their substitution in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$$

and subsequent integration leads to a complete integral of the given equation.

Ex. 1. *Find a complete integral of* $p_1^3 + p_2^2 + p_3 = 1$.

Sol. Let the given equation be rewritten as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0. \tag{1}$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or

$$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}, \quad \text{using (1)}.$$

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$. Here

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1 \tag{2}$$

and

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2. \qquad (3)$$

Now,

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right)$$

or

$$(F_1, F_2) = \frac{\partial F_1}{\partial x_1} \frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1} \frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2} \frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2} \frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3} \frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3} \frac{\partial F_2}{\partial x_3}.$$

or

$$(F_1, F_2) = (0)(0) - (1)(0) + (0)(1) - (0)(0) + (0)(0) - (0)(0) = 0, \quad \text{by (3) and (4).}$$

Thus, we have verified that for relations (2) and (3), $(F_1, F_2) = 0$. Hence (2) and (3) may be taken as additional equations.
Solving (1), (2) and (3) for p_1, p_2, p_3, $\quad p_1 = a_1, \quad p_2 = a_2, \quad p_3 = 1 - a_1^3 - a_2^2$.
Putting these values in $\quad dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3.$$

Integrating,

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3,$$

which is a complete integral of the given equation containing three arbitrary constants a_1, a_2, and a_3.
Ex. 2. *Find a complete integral of* $p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0$.

Sol. Given

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3(p_1 + p_2) + x_1 + x_2 = 0 \tag{1}$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

$$\Rightarrow \frac{dp_1}{1} = \frac{dx_1}{p_3x_3} = \frac{dp_2}{1} = \frac{dx_2}{-p_3x_3} = \frac{dp_3}{p_3(p_2 + p_3)} = \frac{dx_3}{-x_3(p_1 + p_2)}, \quad \text{by (1)} \tag{2}$$

Taking the two fractions of (2),

$$dp_1 - dp_2 = 0 \quad \Rightarrow \quad p_1 - p_2 = a_1 \tag{3}$$

Let

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - p_2 = a_1$$

Taking the fifth and sixth fractions of (2),

$$\frac{1}{p_3}dp_3 + \frac{1}{x_3}dx_3 = 0 \quad \Rightarrow \quad p_3x_3 = a_3 \tag{4}$$

Let

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3 = a_2$$

Now,

$$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

$$= \left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right) + \left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right) + \left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)$$

$$= (0)(0) - (1)(0) + (0)(0) - (-1)(0) + (0)(x_3) - (0)(p_3) = 0 \quad \text{by (3) and (4)}$$

Thus, we have verified that for the relations (3) and (4), $(F_1, F_2) = 0$.
From (1) and (4),

$$a_2(p_1 + p_2) + x_1 + x_2 = 0 \quad \Rightarrow \quad p_1 + p_2 = -\frac{x_1 + x_2}{a_2} \tag{5}$$

Solving (3) and (5),

$$p_1 = \frac{a_1}{2} - \frac{x_1 + x_2}{2a_2}, \qquad p_2 = -\frac{a_1}{2} - \frac{x_1 + x_2}{2a_2} \tag{6}$$

Again, from (4),

$$p_3 = \frac{a_2}{x_3} \tag{7}$$

Putting the values of p_1, p_2, p_3 given by (6) and (7) in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3,$$

we have

$$dz = \frac{a_1}{2}(dx_1 - dx_2) - \frac{x_1 + x_2}{2a_2}(dx_1 + dx_2) + \frac{a_2}{x_3}dx_3$$

Integrating,

$$z = \frac{a_1}{2}(x_1 - x_2) - \frac{1}{4a_2}(x_1 + x_2)^2 + a_2 \log x_3 + a_3$$

THANK YOU

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