Linear Partial Differential Equation - UNIT 1

Presented by

Dr. Rajshekhar Roy Baruah
Assistant Professor
Department of Mathematical Sciences

DEPARTMENT OF MATHEMATICAL SCIENCES
BODOLAND UNIVERSITY, KOKRAJHAR

Lagrange's Equation:

A quasi–linear partial differential equation of order one is of the form

$$Pp + Qq = R,$$

where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation*.
For example $xp + yzq = zx$ is a Lagrange equation.

Method of Solving $Pp + Qq = R$

Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are functions of x, y, z

Theorem. *The general solution of Lagrange equation*

$$Pp + Qq = R, \tag{1}$$

is

$$\phi(u, v) = 0 \tag{2}$$

where ϕ is an arbitrary function and

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \tag{3}$$

are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{4}$$

Here, c_1 *and* c_2 *are arbitrary constants and at least one of* u, v *must contain* z*. Also* u *and* v *are said to be independent if* u/v *is not merely a constant.*

Working Rule for solving $Pp + Qq = R$ by Lagrange's method

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \tag{1}$$

Step 2. Write down Lagrange's auxiliary equations for (1),

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

Step 3. Solve (2) by using the well known methods. Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Method of Solving $Pp + Qq = R$

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms:

$$\phi(u, v) = 0, \quad u = \phi(v) \quad \text{or} \quad v = \phi(u),$$

ϕ being an arbitrary function.
In what follows we shall discuss four rules for getting two independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Accordingly, we have four types of problems based on $Pp + Qq = R$.

Rule I:
We have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

Ex. 1. Solve $(y^2z/x)p + xzq = y^2$.

Sol. Given,

$$\left(\frac{y^2z}{x}\right)p + xzq = y^2 \tag{1}$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{\left(\frac{y^2 z}{x}\right)} = \frac{dy}{xz} = \frac{dz}{y^2}. \tag{2}$$

Taking the first two fractions of (2), we have

$$x^2\,dx = y^2\,dy$$

or

$$3x^2\,dx - 3y^2\,dy = 0. \tag{3}$$

Integrating (3),

$$x^3 - y^3 = c_1,$$

c_1 being an arbitrary constant (4)

Again, taking the first and the last fractions of (2), we get

$$xy^2dx = y^2dz$$

or

$$2x\,dx - 2z\,dz = 0. \tag{5}$$

Integrating (5),

$$x^2 - z^2 = c_2,$$

c_2 being an arbitrary constant (6)

From (4) and (6), the required general integral is

$$\phi\left(x^3 - y^3,\ x^2 - z^2\right) = 0,$$

ϕ being an arbitrary function.

Problems Based on Rule I

Ex. 2. Solve $(x^2 + 2y^2)p - xyq = xz$

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{x^2 + 2y^2} = \frac{dy}{-xy} = \frac{dz}{xz}. \tag{1}$$

Taking the last two fractions of (2) and re–writing, we get

$$\frac{1}{y}dy + \frac{1}{z}dz = 0$$

so that

$$\log y + \log z = \log c_1 \quad \text{or} \quad yz = c_1. \tag{2}$$

Problems Based on Rule I

Taking the first two fractions of (1), we have

$$\frac{dx}{dy} = \frac{x^2 + 2y^2}{-xy}$$

or

$$2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right)x^2 = -4y. \tag{3}$$

Putting $x^2 = v$ and $2x(dx/dy) = dv/dy$, (3) yields

$$\frac{dv}{dy} + \frac{2}{y}v = -4y,$$

which is a linear equation.

Problems Based on Rule I

Its integrating factor is

$$e^{\int \frac{2}{y} dy} = e^{2 \log y} = y^2$$

and hence its solution is

$$yv^2 = \int \left\{(-4y) \cdot xy^2\right\} dy + c_2$$

or

$$y^2x^2 + y^4 = c_2. \tag{4}$$

From (2) and (4), the required solution is

$$\phi\left(yz,\ y^2x^2 + y^4\right) = 0,$$

ϕ being an arbitrary function.

Problems Based on Rule I

Solve the following partial differential equations

1. $(-a+x)p+(-b+y)q=(-c+z)$ Ans. $\phi\left\{\frac{x-a}{y-b},\frac{y-b}{z-c}\right\}=0$
2. $xp+yq=z$ Ans. $\phi\left(\frac{x}{z},\frac{y}{z}\right)=0$
3. $p+q=1$ Ans. $\phi(x-y,x-z)=0$
4. $x^2p+y^2p=z^2$ Ans. $\phi\left(\frac{1}{x}-\frac{1}{y},\frac{1}{y}-\frac{1}{z}\right)=0$
5. $x^2p+y^2q+z^2=0$ Ans. $\phi\left(\frac{1}{x}-\frac{1}{y},\frac{1}{y}+\frac{1}{z}\right)=0$
6. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\sin x$ Ans. $\phi(x-y,z+\cos x)=0$
7. $yzp+2xq=xy$ Ans. $\phi(x^2-z^2,y^2-4z)=0$
8. $xp+yq=z$ Ans. $\phi\left(\frac{x}{y},\frac{x}{z}\right)=0$

Rule II:
We have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

Suppose that one integral of (1) is known by using Rule I and also that another integral cannot be obtained by using Rule I. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of the first integral should be removed later on.

Ex. 1. Solve $p + 3q = 5z + \tan(y - 3x)$.
Sol. Given

$$p + 3q = 5z + \tan(y - 3x). \tag{1}$$

Problems Based on Rule II

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}. \tag{2}$$

Taking the first two fractions,

$$dy - 3dx = 0. \tag{3}$$

Integrating (3),

$$y - 3x = c_1, \quad c_1 \text{ being an arbitrary constant.} \tag{4}$$

Using (4), from (2) we get

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}. \tag{5}$$

Integrating (5),

$$x - \frac{1}{5}\log(5z + \tan c_1) = \frac{1}{5}c_2, \quad c_2 \text{ being an arbitrary constant.}$$

or

$$5x - \log\big[5z + \tan(y - 3x)\big] = c_2, \text{ using (4).} \tag{6}$$

From (4) and (6), the required general integral is

$$5x - \log\big[5z + \tan(y - 3x)\big] = \phi(y - 3x), \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. Solve $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given

$$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4 \tag{1}$$

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}. \tag{2}$$

Cancelling $z(z^2+xy)$, the first two fractions give

$$\frac{1}{x}dx = -\frac{1}{y}dy \quad \text{or} \quad \frac{1}{x}dx + \frac{1}{y}dy = 0. \tag{3}$$

Integrating (3),

$$\log x + \log y = \log c_1 \quad \text{or} \quad xy = c_1. \tag{4}$$

Problems Based on Rule II

Using (4), from (2) we get

$$\frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4} \quad \text{or} \quad x^3 dx = z(z^2+c_1)dz. \tag{5}$$

Integrating (5),

$$\frac{x^4}{4} - \frac{z^4}{4} - \frac{c_1 z^2}{2} = \frac{c_2}{4} \quad \text{or} \quad x^4 - z^4 - 2c_1 z^2 = c_2, \ \text{using (4)}. \tag{6}$$

From (4) and (6), the required general integral is

$$\phi(xy, x^4 - z^4 - 2xyz^2) = 0,$$

ϕ being an arbitrary function.

Rule III:

We have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction in (1) will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1}. \tag{2}$$

If $PP_1 + QQ_1 + RR_1 = 0$, then we know that the numerator of (2) is also zero. This gives

$$P_1 dx + Q_1 dy + R_1 dz = 0$$

Problems Based on Rule III

which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called *multipliers*. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases, the second integral should be obtained by using rule I or rule II as the case may be.

Ex. 1. *Solve* $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$.

Sol. Given

$$\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy \tag{1}$$

The Lagrange's subsidiary equations of (1) are

$$\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy} \tag{2}$$

Problems Based on Rule III

Choosing x, y, z as multipliers,

$$\text{Each fraction for (2)} = \frac{a\,x\,dx + b\,y\,dy + c\,z\,dz}{xyz\,[(b-c)+(c-a)+(a-b)]} = \frac{a\,x\,dx + b\,y\,dy + c\,z\,dz}{0}$$

$$\Rightarrow a\,x\,dx + b\,y\,dy + c\,z\,dz = 0 \quad \Rightarrow \quad 2a\,x\,dx + 2b\,y\,dy + 2c\,z\,dz = 0$$

Integrating,

$$ax^2 + by^2 + cz^2 = c_1, \quad c_1 \text{ being an arbitrary constant} \tag{3}$$

Again, choosing ax, by, cz as multipliers,

$$\text{Each fraction for (2)} = \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{xyz\,[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{0}$$

$$\Rightarrow a^2x\,dx + b^2y\,dy + c^2z\,dz = 0 \quad \Rightarrow \quad 2a^2x\,dx + 2b^2y\,dy + 2c^2z\,dz = 0$$

Integrating,

$$a^2x^2 + b^2y^2 + c^2z^2 = c_2, \quad c_2 \text{ being an arbitrary constant} \tag{4}$$

From (3) and (4), the required general solution is given by

$$\phi\left(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2\right) = 0,$$

where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(x+y)p + z(x-y)q = x^2 + y^2.$

Sol. Given

$$z(x+y)p + z(x-y)q = x^2 + y^2 \tag{1}$$

Problems Based on Rule III

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \tag{2}$$

Choosing $x, -y, -z$ as multipliers, each fraction becomes

$$\frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$$\therefore \quad x\,dx - y\,dy - z\,dz = 0 \quad \text{or} \quad 2x\,dx - 2y\,dy - 2z\,dz = 0.$$

Integrating,

$$x^2 - y^2 - z^2 = c_1, \quad c_1 \text{ being an arbitrary constant} \tag{3}$$

Again, choosing $y, x, -z$ as multipliers, each fraction becomes

$$\frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}.$$

$$\therefore \quad y\,dx + x\,dy - z\,dz = 0 \quad \text{or} \quad 2\,d(xy) - 2z\,dz = 0$$

Integrating,

$$2xy - z^2 = c_2, \quad c_2 \text{ being an arbitrary constant} \tag{4}$$

From (3) and (4), the required general solution is given by

$$\phi\left(x^2 - y^2 - z^2,\ 2xy - z^2\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Problems Based on Rule III

Ex. 3. *Solve* $(mz-ny)p+(nx-lz)q=ly-mx.$

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}. \tag{1}$$

Choosing x, y, z as multipliers, each fraction of (1)

$$=\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{xdx+ydy+zdz}{0}$$

$$\therefore xdx+ydy+zdz=0 \quad \Rightarrow \quad 2xdx+2ydy+2zdz=0$$

Integrating, $x^2+y^2+z^2=c_1$, c_1 being an arbitrary constant. (2)

Problems Based on Rule III

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

$$\therefore ldx + mdy + ndz = 0 \quad \Rightarrow \quad lx + my + nz = c_2. \tag{3}$$

From (2) and (3), the required general solution is given by

$$\phi\left(x^2 + y^2 + z^2,\ lx + my + nz\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 4. Solve $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$.

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \tag{1}$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 = c_1. \tag{2}$$

Choosing $1/x, -1/y, -1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{(y^2 - z^2)/x - (z^2 + x^2)/y - (x^2 + y^2)/z}$$

$$\Rightarrow (1/x)dx - (1/y)dy - (1/z)dz = 0 \quad \Rightarrow \quad \log x - \log y - \log z = \log c_2$$

$$\Rightarrow \frac{x}{yz} = c_2. \tag{3}$$

The required solution is

$$\phi\left(x^2 + y^2 + z^2,\ \frac{x}{yz}\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Rule IV:
We have,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction of (1) will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}. \tag{2}$$

Let, the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$\frac{P_2dx + Q_2dy + R_2dz}{P_2P + Q_2Q + R_2R} \tag{3}$$

is such that its numerator is exact differential of denominator. Fractions (2) and (3) are then combined to give an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 or rule 2 or rule 3.

Ex. 1. *Solve* $(y+z)p + (z+x)q = x+y$.

Sol. Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \tag{1}$$

Choosing $1, -1, 0$ as multipliers,

$$\text{Each fraction of (1)} = \frac{dx - dy}{(y+z)-(z+x)} = \frac{d(x-y)}{-(x-y)} \tag{2}$$

Again, choosing $0, 1, -1$ as multipliers,

$$\text{Each fraction of (1)} = \frac{dy - dz}{(z+x)-(x+y)} = \frac{d(y-z)}{-(y-z)} \tag{3}$$

Finally, choosing $1, 1, 1$ as multipliers,

$$\text{Each fraction of (1)} = \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \tag{4}$$

Problems Based on Rule IV

From (2), (3) and (4)

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}. \tag{5}$$

Taking the first two fractions of (5),

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating,

$$\log(x-y) = \log(y-z) + \log c_1, \quad c_1 \text{ being an arbitrary constant.}$$

$$\log\frac{(x-y)}{(y-z)} = \log c_1 \quad \text{or} \quad \frac{(x-y)}{(y-z)} = c_1. \tag{6}$$

Taking the first and the third fractions of (5),

$$2\frac{d(x-y)}{(x-y)} + \frac{d(x+y+z)}{x+y+z} = 0$$

Integrating,

$$2\log(x-y) + \log(x+y+z) = \log c_2$$

$$\text{or,} \quad (x-y)^2(x+y+z) = c_2. \tag{7}$$

From (6) and (7), the required general solution is

$$\phi\left((x-y)^2(x+y+z),\ \frac{(x-y)}{(y-z)}\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$

Sol. Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}. \qquad (1)$$

Taking the first two fractions of (1),

$$x^2dx = -y^2dy \quad \text{or} \quad 3x^2dx + 3y^2dy = 0.$$

Integrating,

$$x^3 + y^3 = c_1, \quad c_1 \text{ being an arbitrary constant.} \qquad (2)$$

Choosing $1, -1, 0$ as multipliers,

$$\text{Each fraction of (1)} = \frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dx - dy}{(x-y)(x^2+y^2)}. \tag{3}$$

Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx - dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating,

$$\log(x-y) - \log z = \log c_2 \quad \text{or} \quad \frac{(x-y)}{z} = c_2. \tag{4}$$

From (3) and (4), solution is

$$\phi\left(x^3+y^3,\ \frac{x-y}{z}\right)=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 3. Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$.

Sol. Here the Lagrange's auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \tag{1}$$

Choosing $1,-1,0$ and $0,1,-1$ as multipliers in turn,

$$\text{Each fraction of (1)}=\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{y^2-z^2+x(y-z)}$$

Problems Based on Rule IV

so that

$$\frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(y+z+x)}$$

or

$$\frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

Integrating,

$$\log(x-y) - \log(y-z) = \log c_2$$

or

$$(x-y)(y-z) = c_1 \tag{2}$$

Problems Based on Rule IV

Choosing x, y, z as multipliers,

$$\text{Each fraction of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} \tag{3}$$

Again, choosing $1, 1, 1$ as multipliers,

$$\text{Each fraction of (1)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} \tag{4}$$

From (3) and (4),

$$\frac{x\,dx + y\,dy + z\,dz}{x + y + z} = dx + dy + dz$$

or

$$2(x+y+z)\,d(x+y+z) - (2x\,dx + 2y\,dy + 2z\,dz) = 0$$

Problems Based on Rule IV

Integrating,

$$(x+y+z)^2 - (x^2+y^2+z^2) = 2c_2$$

or

$$(x^2+y^2+z^2+2xy+2yz+2zx) - (x^2+y^2+z^2) = 2c_2$$

or

$$xy+yz+zx = c_2, \quad c_2 \text{ being an arbitrary constant.} \tag{5}$$

From (2) and (5), the required general solution is given by

$$\phi\left[xy+yz+zx,\ \frac{x-y}{y-z}\right] = 0,$$

ϕ being an arbitrary function.

THANK YOU

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