Partial Differential Equation - UNIT 1

Presented by

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Partial Differential Equation

Definition:

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a *partial differential equation*.

For examples:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \quad \cdots \ (1)$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right) \quad \cdots \ (2)$$

$$z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x \quad \cdots \ (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz \quad \cdots \ (4)$$

$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2} \quad \cdots \ (5)$$

$$y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right) \quad \cdots \ (6)$$

Order of a Partial Differential Equation

The *order* of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

Degree of a Partial Differential Equation

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

Linear and Non-linear Partial Differential Equation

A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

Notations

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad t = \frac{\partial^2 z}{\partial y^2}$$

In case there are n independent variables, then

$$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}, \quad \text{and} \quad p_n = \frac{\partial z}{\partial x_n}$$

Sometimes the partial differentiations are also denoted by making use of suffixes.

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{and so on.}$$

Classification of First Order PDEs

Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations.

Linear equation. A first order equation $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form

$$P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y).$$

For examples,

$$yx^2p + xy^2q = xyz + x^2y^3 \quad \text{and} \quad p + q = z + xy$$

Semi-linear equation. A first order partial differential equation $f(x, y, z, p, q) = 0$ is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form

$$P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$$

Classification of First Order PDEs

Quasi-linear equation. A first order partial differential equation $f(x, y, z, p, q) = 0$ is known as a quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples,

$$x^2 p + y^2 q = xy \quad \text{and} \quad (x^2 - yz)p + (y^2 - zx)q = z^2 - xy$$

Non-linear equation. A first order partial differential equation $f(x, y, z, p, q) = 0$ which does not come under the above three types, is known as a non-linear equation. For examples,

$$p^2 + q^2 = 1, \quad pq = z \quad \text{and} \quad x^2 p^2 + y^2 q^2 = z$$

Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation

$$F(x, y, z, a, b) = 0, \tag{1}$$

where a and b denote arbitrary constants. Let z be regarded as a function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\frac{\partial F}{\partial x} + p\left(\frac{\partial F}{\partial z}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} + q\left(\frac{\partial F}{\partial z}\right) = 0, \tag{2}$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \tag{3}$$

which is a partial differential equation of the first order.
In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems:

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$$F(x, y, z, a, b) = 0, \quad \frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0.$$

The equation free from a and b will be the required partial differential equation.
Three situations may arise:

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*
For example, let us consider

$$z = ax + y \tag{1}$$

where a is the only arbitrary constant and x, y are two independent variables.
Differentiating (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = a \tag{2}$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = 1 \tag{3}$$

Eliminating a between (1) and (2) yields

$$z = x\left(\frac{\partial z}{\partial x}\right) + y \tag{4}$$

Since (3) does not contain an arbitrary constant, it is also a partial differential equation. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: Eliminate a and b from

$$az + b = a^2 x + y \tag{1}$$

Differentiating (1) partially with respect to x and y,

$$a\left(\frac{\partial z}{\partial x}\right) = a^2 \tag{2}$$

$$a\left(\frac{\partial z}{\partial y}\right) = 1 \tag{3}$$

Eliminating a from (2) and (3),

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1 \tag{4}$$

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from*

$$z = ax + by + cxy \tag{1}$$

Origin of PDEs

Differentiating (1) partially with respect to x and y, we have

$$\frac{\partial z}{\partial x} = a + cy \tag{2}$$

$$\frac{\partial z}{\partial y} = b + cx \tag{3}$$

Again differentiating (2) and (3),

$$\frac{\partial^2 z}{\partial x^2} = 0, \frac{\partial^2 z}{\partial y^2} = 0 \tag{4}$$

$$\frac{\partial^2 z}{\partial x \partial y} = c \tag{5}$$

Now, using (2) and (3),

$$x\left(\frac{\partial z}{\partial x}\right) = ax + cxy$$

$$y\left(\frac{\partial z}{\partial y}\right) = by + cxy$$

$$\therefore \quad x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = ax + by + 2cxy = z + cxy \Rightarrow$$

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) \tag{6}$$

Thus, we get three partial differential equations given by (4), (5), and (6), all of order two.

Ex. 1. *Find a partial differential equation by eliminating a and b from*

$$z = ax + by + a^2 + b^2$$

Sol. Given:

$$z = ax + by + a^2 + b^2 \tag{1}$$

Differentiating (1) partially with respect to x and y, we get:

$$\frac{\partial z}{\partial x} = a \quad \text{and} \quad \frac{\partial z}{\partial y} = b$$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from $z = (x-a)^2 + (y-b)^2$ to form the partial differential equation.*

Sol. Given

$$z = (x-a)^2 + (y-b)^2. \tag{1}$$

Differentiating (1) partially with respect to a and b, we get

$$\frac{\partial z}{\partial a} = 2(x-a) \quad \text{and} \quad \frac{\partial z}{\partial b} = 2(y-b).$$

Squaring and adding these equations, we have

$$\left(\frac{\partial z}{\partial a}\right)^2 + \left(\frac{\partial z}{\partial b}\right)^2 = 4(x-a)^2 + 4(y-b)^2 = 4\left[(x-a)^2 + (y-b)^2\right]$$

or

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z, \quad \text{using (1).}$$

Q. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations:*

1. $z = a(x+y) + b$.
2. $z = ax + by + ab$.
3. $z = ax + a^2y^2 + b$.
4. $z = (x+a)(y+b)$.

Origin of PDEs

Ex. 3. *Find a partial differential equation by eliminating a, b, c, from $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.*

Sol. Given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2x + a^2z\frac{\partial z}{\partial x} = 0 \tag{2}$$

and

$$\frac{2y}{b^2} + \frac{2x}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2y + b^2z\frac{\partial z}{\partial y} = 0 \tag{3}$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z\frac{\partial^2 z}{\partial x^2} = 0 \tag{4}$$

$$c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2 z\frac{\partial^2 z}{\partial y^2} = 0 \tag{5}$$

From (2),

$$c^2 = -\frac{a^2 z}{x}\left(\frac{\partial z}{\partial x}\right) \tag{6}$$

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \tag{7}$$

Similarly, from (3) and (5),

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \tag{8}$$

Differentiating (2) partially with respect to y,

$$0 + a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + z\frac{\partial^2 z}{\partial x \partial y}\right\} = 0$$

or

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + z\frac{\partial^2 z}{\partial x \partial y} = 0 \tag{9}$$

(7), (8) and (9) are three possible forms of the required partial differential equations.

Rule II. Derivation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

Proof. Given

$$\phi(u, v) = 0 \tag{1}$$

We treat z as dependent variable and x and y as independent variables so that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial y}{\partial x} = 0, \quad \frac{\partial x}{\partial y} = 0.$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

or

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}}=-\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}} \tag{3}$$

Similarly, differentiating (1) partially with respect to y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

or

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}}=-\frac{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}} \tag{4}$$

Eliminating ϕ with the help of (3) and (4), we get

$$\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}}$$

or

$$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or

$$Pp + Qq = R \tag{5}$$

where

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}, \quad Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}, \quad R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Thus we obtain a linear partial differential equation of first order and of first degree in p and q.

Ex. 1. Form a partial differential equation by eliminating the arbitrary function ϕ from

$$\phi(x+y+z,\, x^2+y^2-z^2) = 0.$$

What is the order of this partial differential equation?

Sol. Given

$$\phi(x+y+z,\, x^2+y^2-z^2) = 0 \tag{1}$$

Let,

$$u = x+y+z, \quad v = x^2+y^2-z^2 \tag{2}$$

Origin of PDEs

Then (1) becomes

$$\phi(u, v) = 0. \tag{3}$$

Differentiating (3) w.r.t. x partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0. \tag{4}$$

From (2),

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial z} = 1, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial z} = -2z,$$

$$\frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial y} = 2y. \tag{5}$$

From (4) and (5),

$$(\phi_u)(1+p) + 2(\phi_v)(x - pz) = 0, \quad \text{or} \quad \frac{\phi_u}{\phi_v} = -\frac{2(x-pz)}{1+p}. \tag{6}$$

Again, differentiating (3) w.r.t. y partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0, \tag{7}$$

or

$$(\phi_u)(1+q) + 2(\phi_v)(y - qz) = 0, \quad \text{by (5)}$$

or

$$\frac{\phi_u}{\phi_v} = -\frac{2(y-qz)}{1+q}.$$

Origin of PDEs

Eliminating ϕ from (6) and (7), we obtain

$$\frac{x - pz}{1 + p} = \frac{y - qz}{1 + q}$$

or

$$1 + q)(x - pz) = (1 + p)(y - qz)$$

or

$$y + zp - (x + z)q = x - y,$$

which is the desired partial differential equation of first order.

Ex. 2. *Form a partial differential equation by eliminating the arbitrary function f from the equation $x + y + z = f(x^2 + y^2 + z^2)$.*

Sol. Given

$$x + y + z = f(x^2 + y^2 + z^2) \tag{1}$$

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$$1 + p = f'(x^2 + y^2 + z^2) \cdot (2x + 2zp) \tag{2}$$

and

$$1 + q = f'(x^2 + y^2 + z^2) \cdot (2y + 2zq) \tag{3}$$

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$$\frac{1+p}{2x+2zp} = \frac{1+q}{2y+2zq} \quad \text{or} \quad (1+p)(y+zq) = (1+q)(x+zp)$$

or

$$(y-z)p+(z-x)q=x-y$$

which is the required partial differential equation.

Ex. 3. *Eliminate the arbitrary functions f and F from $y=f(x-at)+F(x+at)$.*

Sol. Given

$$y=f(x-at)+F(x+at) \tag{1}$$

From (1),

$$\frac{\partial y}{\partial x}=f'(x-at)+F'(x+at)$$

and hence

$$\frac{\partial^2 y}{\partial x^2}=f''(x-at)+F''(x+at) \tag{2}$$

Origin of PDEs

Also,

$$\frac{\partial y}{\partial t} = f'(x-at)(-a) + F'(x+at)(a)$$

and hence

$$\frac{\partial^2 y}{\partial t^2} = f''(x-at)(-a)^2 + F''(x+at)(a)^2$$

or

$$\frac{\partial^2 y}{\partial t^2} = a^2\left[f''(x-at) + F''(x+at)\right] \tag{3}$$

Then, from (2) and (3), we get

$$\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Ex. 4. *Form a partial differential equation by eliminating the arbitrary function ϕ from*

$$\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0.$$

Sol. Given

$$\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0 \tag{1}$$

Let

$$u = x^2 + y^2 + z^2, \quad v = z^2 - 2xy.$$

Then (1) becomes

$$\phi(u, v) = 0 \tag{3}$$

Origin of PDEs

Differentiating (3) partially w.r.t. x, we get

$$\phi_u \frac{\partial u}{\partial x} + \phi_v \frac{\partial v}{\partial x} = 0$$

or

$$\phi_u(2x + 2zp) + \phi_v(2zp - 2y) = 0 \tag{4}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.
From (2), we have

$$\frac{\partial u}{\partial x} = 2x + 2zp, \quad \frac{\partial v}{\partial x} = 2zp - 2y.$$

Using (5), (4) reduces to

$$\phi_u(2x + 2zp) + \phi_v(2zp - 2y) = 0 \tag{6}$$

Again, differentiating (3) partially w.r.t. y, we get

$$\phi_u(2y + 2zq) + \phi_v(2zq - 2x) = 0$$

or

$$\phi_u(2y + 2zq) + \phi_v(2zq - 2x) = 0 \tag{7}$$

Dividing (6) by (7),

$$\frac{(2x + 2zp)}{(2y + 2zq)} = \frac{(2zp - 2y)}{(2zq - 2x)}$$

which simplifies to

$$p(x + zp) - q(y + zq) = y - x$$

or

$$(p - q)z = y - x.$$

Ex. 5. *Equation of any cone with vertex at $P(a,b,c)$ is of the form*

$$f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0.$$

Find the differential equation of the cone.

Sol. Let

$$\frac{x-a}{z-c} = u \quad \text{and} \quad \frac{y-b}{z-c} = v \tag{1}$$

Then, the equation of the given cone becomes

$$f(u,v) = 0 \tag{2}$$

Differentiating (2) partially with respect to x, we have

$$\begin{aligned}
&\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0\\
\Rightarrow&\frac{\partial f}{\partial u}\left(\frac{1-0}{z-c}-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right)=0, \quad \text{using (1)}\\
\Rightarrow&\frac{\partial f}{\partial u}\left(\frac{1}{z-c}-p\frac{x-a}{(z-c)^2}\right)-\frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right)=0, \quad \text{where} \quad p=\frac{\partial z}{\partial x} \qquad (3)
\end{aligned}$$

Differentiating (2) partially with respect to y, we have

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$$
\begin{aligned}
&\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0\\
\Rightarrow&\frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{1-0}{z-c}-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right)=0, \quad \text{using (1)}\\
\Rightarrow&-\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z-c}-q\frac{y-b}{(z-c)^2}\right)=0, \quad \text{where} \quad q=\frac{\partial z}{\partial y} \qquad (4)
\end{aligned}
$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we have

$$
\begin{vmatrix} \frac{1}{z-c}-p\frac{x-a}{(z-c)^2} & -p\frac{y-b}{(z-c)^2} \\ -q\frac{x-a}{(z-c)^2} & \frac{1}{z-c}-q\frac{y-b}{(z-c)^2} \end{vmatrix}=0
$$

or

$$\begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix} = 0$$

or

$$\{z-c-p(x-a)\}\{z-c-q(y-b)\} - pq(x-a)(y-b) = 0$$

or

$$(z-c)^2 - p(x-a)(z-c) - q(y-b)(z-c) = 0$$

or

$$(x-a)p + (y-b)q = z-c$$

which is the required partial differential equation of the given cone.

THANK YOU

Vist the website for notes

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