Schwarzschild's Exterior Solution

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Solution of Einstein's Gravitational Equations in Empty Space

Schwarzschild's Exterior Solution for the Gravitational Field of an Isolated Particle

Einstein's original field equations representing the law of gravitation in empty space are given by:

$$R_{\mu\nu} = 0, \tag{1}$$

However, if we include the cosmological constant Λ, Einstein's law of gravitation in empty space is modified as:

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2}$$

The solution to the above equations is simply to find the line element for an interval in empty space surrounding a gravitating point particle, which eventually corresponds to the field of an isolated particle that is always at rest at its origin.

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Schwarzschild first proposed the solution.
In the absence of any mass, the space-time would be flat so that the line element in spherical polar coordinates would be expressed as:

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2. \tag{3}$$

The presence of the mass point would modify the line element. However, since mass is static and isolated, the line element would be spatially spherically symmetric about the point mass and is static. The most general form of such a line element may be expressed as:

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^{\nu} dt^2, \tag{4}$$

where λ and ν are functions of r only; since for a spherically symmetric isolated particle, the field will depend on r alone and not on θ and ϕ.

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Since the gravitational field (i.e., the disturbance from flat-space time) due to a particle diminishes indefinitely as we go to an infinite distance, therefore the line element (4) must reduce to Galilean line element (3) at an infinite distance from the particle. Hence λ and ν must tend to zero as r tends to infinity.
The line element in general relativity is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{5}$$

Here the coordinates are:

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \quad \text{and} \quad x^4 = t. \tag{6}$$

Comparing (4) and (5) with the help of (6), we get:

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$$g_{\mu\nu} = \begin{bmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & e^{\nu} \end{bmatrix}. \tag{7}$$

Then the determinant of $g_{\mu\nu}$ is:

$$g \equiv |g_{\mu\nu}| = -e^{\lambda}(-r^2)(-r^2\sin^2\theta)e^{\nu} = -e^{\lambda+\nu}r^4\sin^2\theta. \tag{8}$$

Using

$$g^{\mu\nu} = \frac{\text{Cofactor of } g^{\mu\nu} \text{ in g}}{g} \tag{9}$$

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$$g^{\mu\nu} = \begin{bmatrix} -e^{-\lambda} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} & 0 \\ 0 & 0 & 0 & e^{-\nu} \end{bmatrix}. \tag{10}$$

If μ, ν, σ are different suffixes, we can easily get the following possible cases

$$\begin{aligned} \Gamma^{\mu}_{\mu\mu} &= \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial(\log g_{\mu\mu})}{\partial x^{\mu}} & \Gamma^{\nu}_{\mu\mu} &= \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x^{\nu}} \\ \Gamma^{\nu}_{\mu\nu} &= \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial(\log g_{\nu\nu})}{\partial x^{\mu}} & \Gamma^{\sigma}_{\mu\nu} &= 0 \end{aligned} \tag{11}$$

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Nine independent non-vanishing Christoffel symbols are:

$$\begin{aligned}
&\Gamma^1_{11} = \frac{1}{2}\frac{\partial\lambda}{\partial r}, \quad \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^4_{14} = \frac{1}{2}\frac{\partial\nu}{\partial r} \\
&\Gamma^1_{22} = -re^{-\lambda}, \quad \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, \quad \Gamma^1_{44} = \frac{1}{2}e^{\nu-\lambda}\frac{\partial\nu}{\partial r} \\
&\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta
\end{aligned} \tag{12}$$

The Ricci tensor components are given by:

$$\begin{aligned}
R_{\mu\nu} &= \frac{\partial}{\partial x^\nu}\Gamma^\beta_{\mu\beta} - \frac{\partial}{\partial x^\beta}\Gamma^\beta_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} \\
&= \frac{\partial^2 \log\sqrt{|g|}}{\partial x^\mu \partial x^\nu} - \frac{\partial}{\partial x^\beta}\Gamma^\beta_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\frac{\partial \log\sqrt{|g|}}{\partial x^\alpha}
\end{aligned} \tag{13}$$

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For R_{11}:

$$R_{11} = \frac{\partial^2 \log\sqrt{|g|}}{\partial x^1 \partial x^1} - \frac{\partial}{\partial x^\beta}\Gamma^\beta_{11} + \Gamma^\alpha_{1\beta}\Gamma^\beta_{\alpha 1} - \Gamma^\alpha_{11}\frac{\partial \log\sqrt{|g|}}{\partial x^\alpha}$$
$$= \frac{\partial^2 \log\sqrt{|g|}}{\partial r^2} - \frac{\partial}{\partial r}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^2_{21} + \Gamma^3_{13}\Gamma^3_{31} + \Gamma^4_{14}\Gamma^4_{41} - \Gamma^1_{11}\frac{\partial \log\sqrt{|g|}}{\partial r} \quad (14)$$

As,

$$|g| = e^{\lambda+\nu} r^4 \sin^2\theta \quad \text{i.e.} \quad \sqrt{|g|} = e^{(\lambda+\nu)/2} r^2 \sin\theta \quad (15)$$

Therefore,

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$$
\begin{aligned}
R_{11} &= \frac{\partial^2}{\partial r^2}\left(\frac{\lambda+\nu}{2} + 2\log r + \log\sin\theta\right) - \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial\lambda}{\partial r}\right) + \left(\frac{1}{2}\frac{\partial\lambda}{\partial r}\right)^2 + \frac{1}{r^2} \\
&\quad + \frac{1}{r^2} + \left(\frac{1}{2}\frac{\partial v}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial\lambda}{\partial r}\frac{\partial}{\partial r}\left(\frac{\lambda+\nu}{2} + 2\log r + \log\sin\theta\right) \\
&= \frac{1}{2}\frac{\partial^2\nu}{\partial r^2} + \frac{1}{4}\left(\frac{\partial\nu}{\partial r^2}\right)^2 - \frac{1}{4}\frac{\partial\lambda}{\partial r}\frac{\partial\nu}{\partial r} - \frac{1}{r}\frac{\partial\lambda}{\partial r} \qquad (16)
\end{aligned}
$$

Similarly,

$$
R_{22} = e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial\nu}{\partial r} - \frac{1}{2}r\frac{\partial\lambda}{\partial r}\right) - 1 \qquad (17)
$$

$$
R_{33} = \left[e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial\nu}{\partial r} - \frac{1}{2}r\frac{\partial\lambda}{\partial r}\right) - 1\right]\sin^2\theta = R_{22}\sin^2\theta \qquad (18)
$$

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$$R_{44} = -\frac{1}{2}e^{-\lambda}\left[\frac{\partial^2\nu}{\partial r^2} + \frac{1}{2}\left(\frac{\partial\nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial\nu}{\partial r}\frac{\partial\lambda}{\partial r} + \frac{2}{r}\frac{\partial\nu}{\partial r}\right] \tag{19}$$

Further, for the above line element, all the off-diagonal components of $R_{\mu\nu}$ are identically zero. Hence, Einstein's field equations for empty space are:

$$R_{\mu\nu} = 0 \tag{20}$$

i.e.,

$$R_{11} = \frac{1}{2}\frac{\partial^2\nu}{\partial r^2} + \frac{1}{4}\left(\frac{\partial\nu}{\partial r}\right)^2 - \frac{1}{4}\frac{\partial\nu}{\partial r}\frac{\partial\lambda}{\partial r} - \frac{1}{2r}\frac{\partial\lambda}{\partial r} = 0 \tag{21}$$

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$$R_{22} = e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1 = 0 \tag{22}$$

$$R_{33} = R_{22}\sin^2\theta = \left[e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1\right]\sin^2\theta = 0 \tag{23}$$

$$R_{44} = -\frac{1}{2}e^{\nu-\lambda}\left[\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{2}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} + \frac{2}{r}\frac{\partial \nu}{\partial r}\right] = 0 \tag{24}$$

Obviously, equation (23) is a mere repetition of equation (22). Thus, the only Einstein's field equations for empty space to be satisfied by λ and ν are:

$$\frac{1}{2}\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{4}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{4}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} - \frac{1}{r}\frac{\partial \lambda}{\partial r} = 0 \tag{25}$$

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$$e^{-\lambda}\left(1+\frac{1}{2}r\frac{\partial\nu}{\partial r}-\frac{1}{2}r\frac{\partial\lambda}{\partial r}\right)-1=0 \tag{26}$$

$$\frac{1}{2}e^{\nu-\lambda}\left[\frac{\partial^2\nu}{\partial r^2}+\frac{1}{2}\left(\frac{\partial\nu}{\partial r}\right)^2-\frac{1}{2}\frac{\partial\nu}{\partial r}\frac{\partial\lambda}{\partial r}+\frac{2}{r}\frac{\partial\nu}{\partial r}\right]=0 \tag{27}$$

Dividing equation (27) by $e^{\nu-\lambda}$ and then subtracting equation (25) from the resulting equation, we get:

$$\frac{1}{r}\frac{\partial\nu}{\partial r}+\frac{1}{r}\frac{\partial\lambda}{\partial r}=0$$

$$\frac{\partial\nu}{\partial r}+\frac{\partial\lambda}{\partial r}=0 \tag{28}$$

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Integrating, we get:

$$\nu + \lambda = A \tag{29}$$

where A is a constant of integration which may be set equal to zero, without any loss of generality, since at $r = \infty$, $\lambda = 0$ and $\nu = 0$. Hence

$$\lambda = -\nu \tag{30}$$

Substituting this in eqn. (17), we get

$$e^{\nu}\left(1 + r\frac{\partial \nu}{\partial r}\right) = 1$$

$$\text{i.e.} \quad \frac{\partial}{\partial r}\left(re^{\nu}\right) = 1$$

Integrating, we get

$$re^{\nu} = r + B,$$

where B is a constant of integration.
i.e.,

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r}$$

where we have put $B = -2m$. This is done in order to facilitate the physical interpretation of m as the mass of the gravitating particle. Thus, the line element due to a static, isolated gravitating mass point, from eqn. (4), is

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$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2$$

This solution was first obtained by Schwarzschild and hence is known as the **Schwarzschild line element**. It is obvious that in the limit $r \to \infty$, the Schwarzschild line element reduces to the line element of flat space-time of special relativity.

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