FLUID DYNAMICS

Velocity Potential

Velocity Potential

Let $\vec{q}$ be the fluid velocity at any instant t then the equations of the streamline at that instant is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

These curves cut the surfaces

$$udx + vdy + wdz = 0$$

orthogonally.

Let us consider a scalar function $\phi(x, y, z, t)$ at that instant, uniform throughout the entire field such that

$$udx + vdy + wdz = -d\phi$$

$$=> udx + vdy + wdz = -\frac{\partial \phi}{\partial x}dx - \frac{\partial \phi}{\partial y}dy - \frac{\partial \phi}{\partial z}dz$$

Therefore,

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}$$

$$=> \vec{q} = -\nabla\phi = -\text{grad}\phi$$

where ϕ is termed as the velocity potential for the field $\vec{q}$. The negative sign is taken as the matter of convention. This shows that ϕ decreases with an increase in the value of x, y or z i.e. the flow is always in the direction of decreasing ϕ.

It ensures that the flow takes place from the higher to lower potentials. The velocity potential is a scalar function of space and time.

The necessary and sufficient condition for $\vec{q} = -\nabla\phi$ to hold is

$$\nabla \times \vec{q} = 0$$

$$=> \hat{\imath}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

If the above relation exists then the flow is said to be irrotational. In other words when the motion is irrotational the velocity vector is the gradient of a scalar function $\phi(x, y, z, t)$.

Remark 1. The surfaces $\phi(x, y, z, t) =$ constant are called the equipotentials. The streamlines

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

are cut at right angles by the surfaces given by the equation

$$udx + vdy + wdz = 0$$

and the condition for the existence of such orthogonal surfaces is the condition that $udx + vdy + wdz = 0$ may posses a solution of the form $\phi(x, y, z, t) =$ constant at the considered instant t, the analytical condition being

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}$$

When the velocity potential exists, then the above equation holds. Therefore,

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right) = -\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y} = 0$$

$$=> \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$

Similarly,

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Remark 2. When $\nabla \times \vec{q} = 0$ holds, the flow is known as the potential kind. It is also known as irrotational. For such flow the field of $\vec{q}$ is conservative.

Remark 3. The equation of continuity of an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Let the fluid move irrotationally. Then the velocity potential ϕ exists such that

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}$$

Therefore we get,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Thus ϕ is a harmonic function satisfying the Laplace equation $\nabla^2\phi = 0$, where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Vorticity Vector

Let $\vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ be the fluid velocity such that $\vec{q} \neq 0$. Then the vector $\vec{\Omega} = curl\ \vec{q}$ is called the *vorticity vector*.

Let, Ω_x, Ω_y, Ω_z be the components of $\vec{\Omega}$ in cartesian coordinates. Then

$$\Omega_x\hat{\imath} + \Omega_y\hat{\jmath} + \Omega_z\hat{k} = \hat{\imath}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

so that,

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Remark 1: In two dimensional cartesian coordinates, the vorticity is given by

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Remark 2: In two dimensional polar coordinates, the vorticity is given by

$$\Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Vortex Lines

A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\vec{\Omega}$.

Let, $\vec{\Omega} = \Omega_x\hat{\imath} + \Omega_y\hat{\jmath} + \Omega_z\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be the position vector of a point P on a vortex line. Then $\vec{\Omega}$ is parallel to $\mathrm{d}\vec{r}$ at P on the vortex line. Hence the equation of vortex lines is given by

$$\vec{\Omega} \times \mathrm{d}\vec{r} = 0$$

$$=> \left(\Omega_x\hat{\imath} + \Omega_y\hat{\jmath} + \Omega_z\hat{k}\right) \times \left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = 0$$

$$=> \left(\Omega_y dz - \Omega_z dy\right)\hat{\imath} + \left(\Omega_z dx - \Omega_x dz\right)\hat{\jmath} + \left(\Omega_x dy - \Omega_y dx\right)\hat{k} = 0$$

$$=> \Omega_y dz - \Omega_z dy = 0,\ \Omega_z dx - \Omega_x dz = 0,\ \Omega_x dy - \Omega_y dx = 0$$

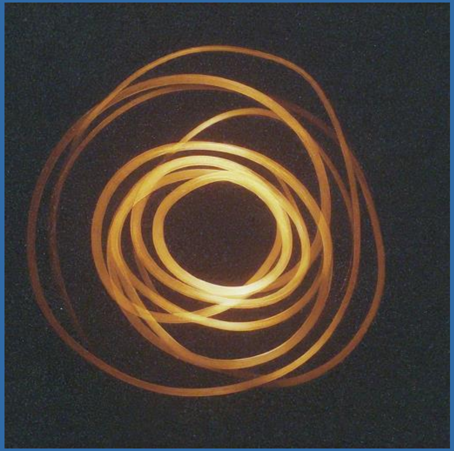
$$=> \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

which gives the equations of vortex lines.

Vortex Tube and Vortex Filament

If we draw the vortex lines from each point of a closed curve in the fluid we obtain a tube called the *vortex tube*.

A vortex tube of infinitesimal cross-section is known as *vortex filament*.



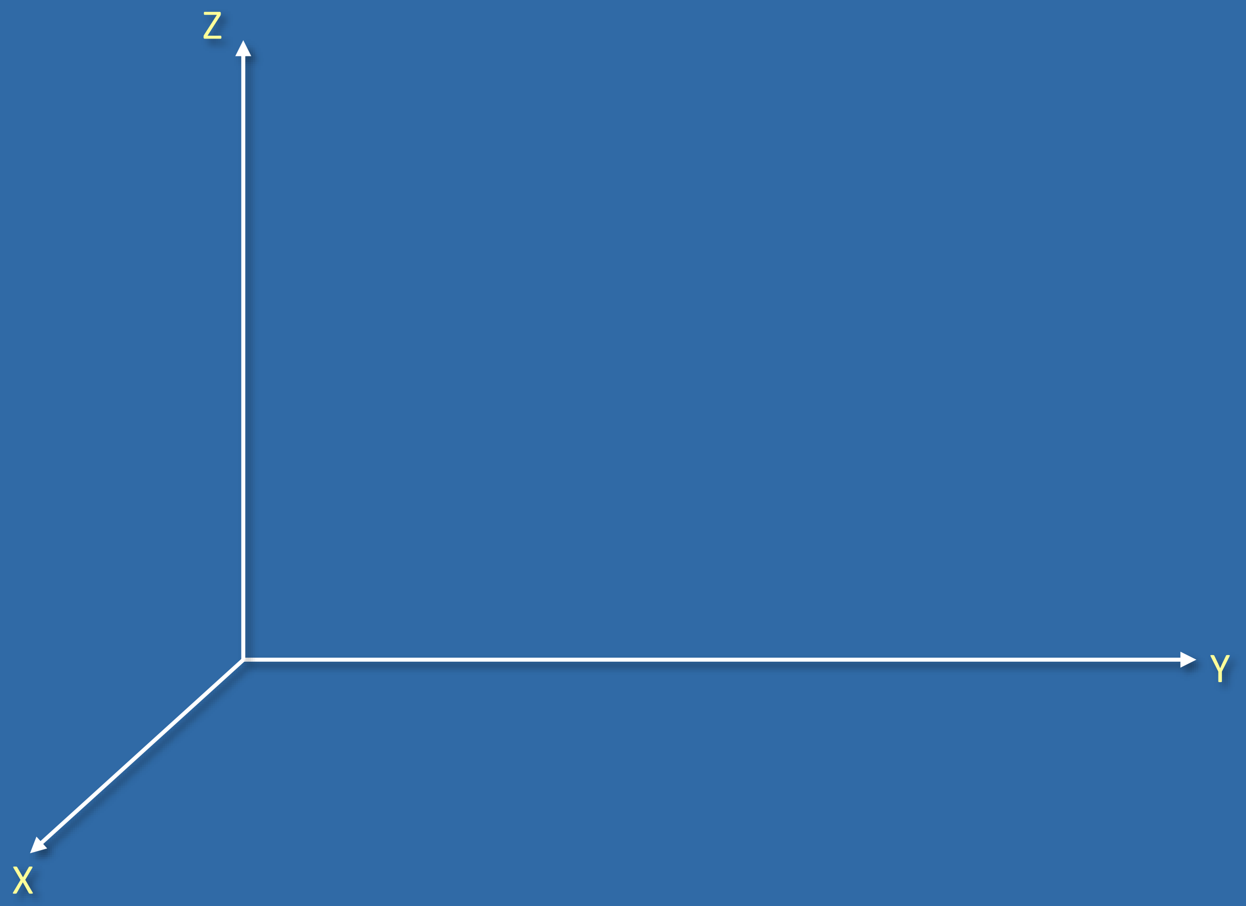
Rotational and Irrotational Motion

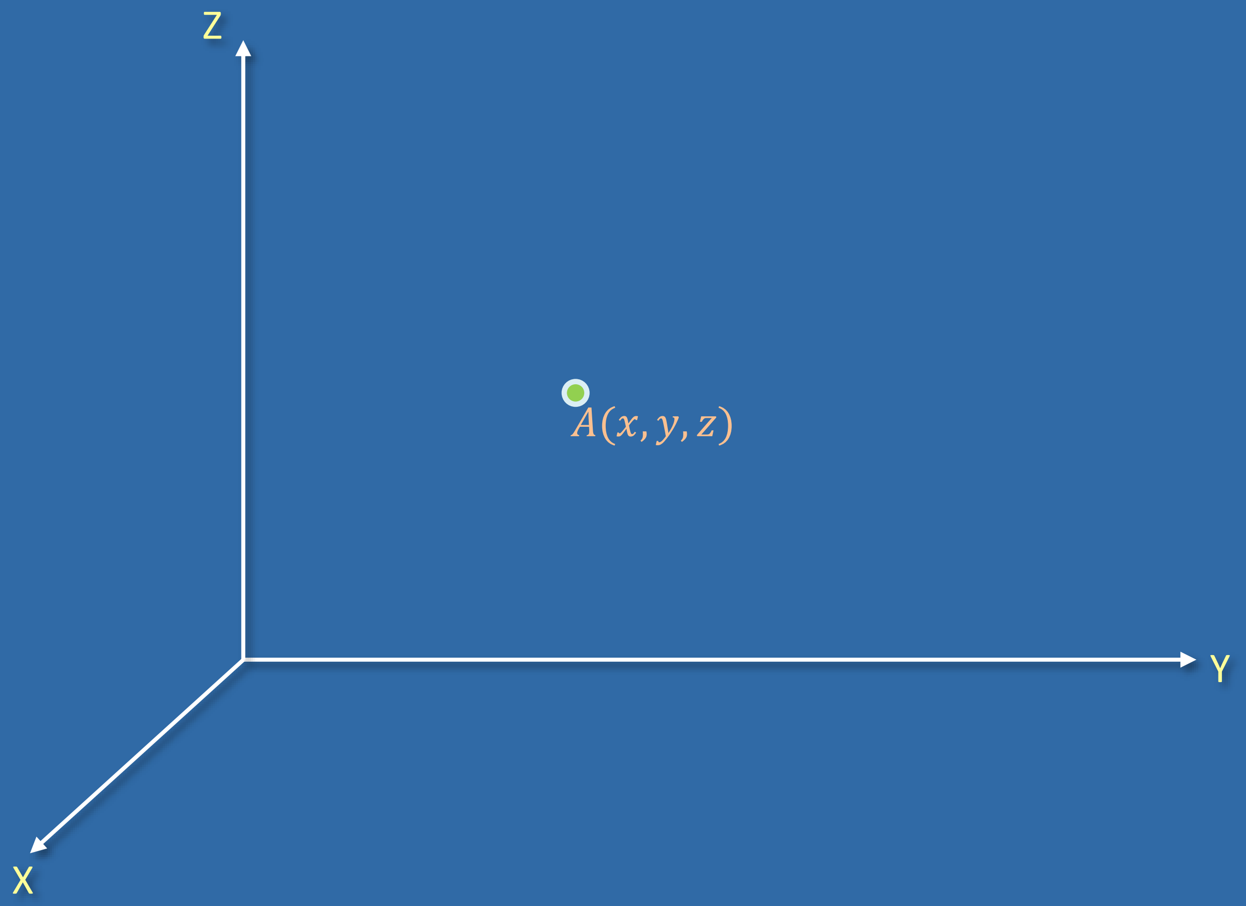
The motion of a fluid is said to be *irrotational* when the vorticity vector $\vec{\Omega}$ of every fluid particle is zero. When the vorticity vector is different from zero, the motion is said to be *rotational*.

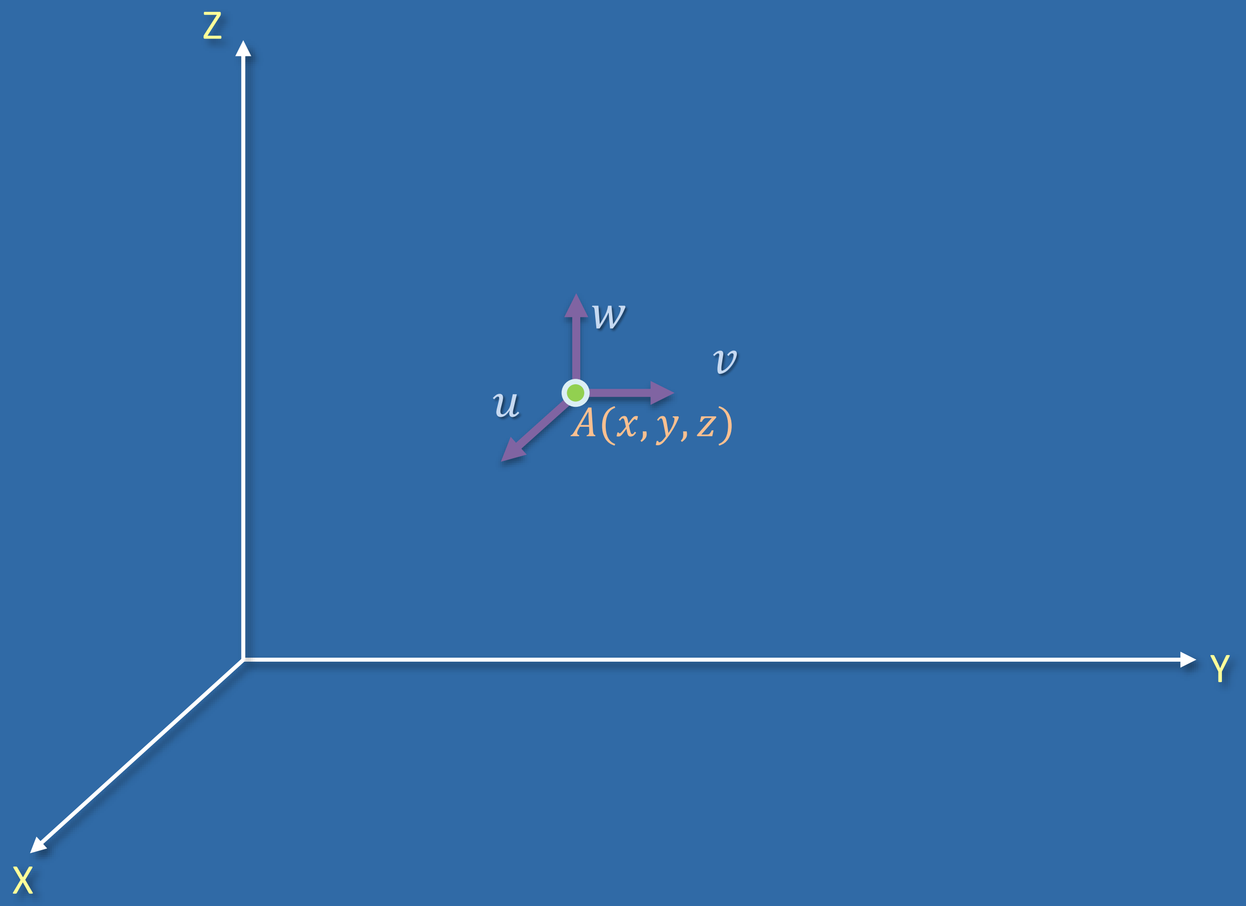
When the motion is *irrotational* i.e. when $\text{curl}\vec{q} = 0$, then $\vec{q}$ must be of the form $(-grad\phi)$ for some scalar point function ϕ (say) because $curl\ grad\phi = 0$. Thus velocity potential exists whenever the fluid motion is irrotational. Again when velocity potential exists, the motion is irrotational because $\vec{q} = -grad\phi => curl\vec{q} = curlgrad\phi = 0$

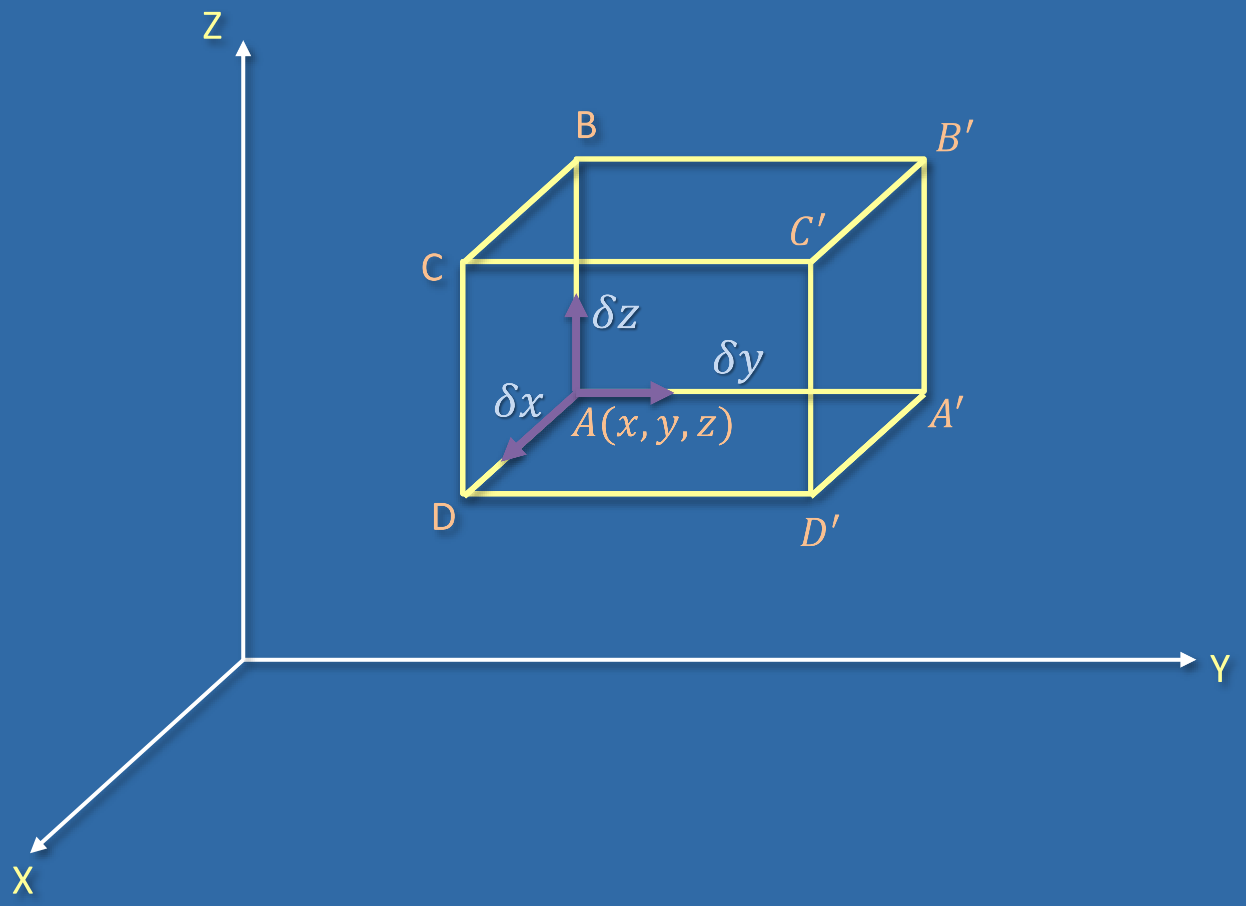
Euler's Equation of Motion

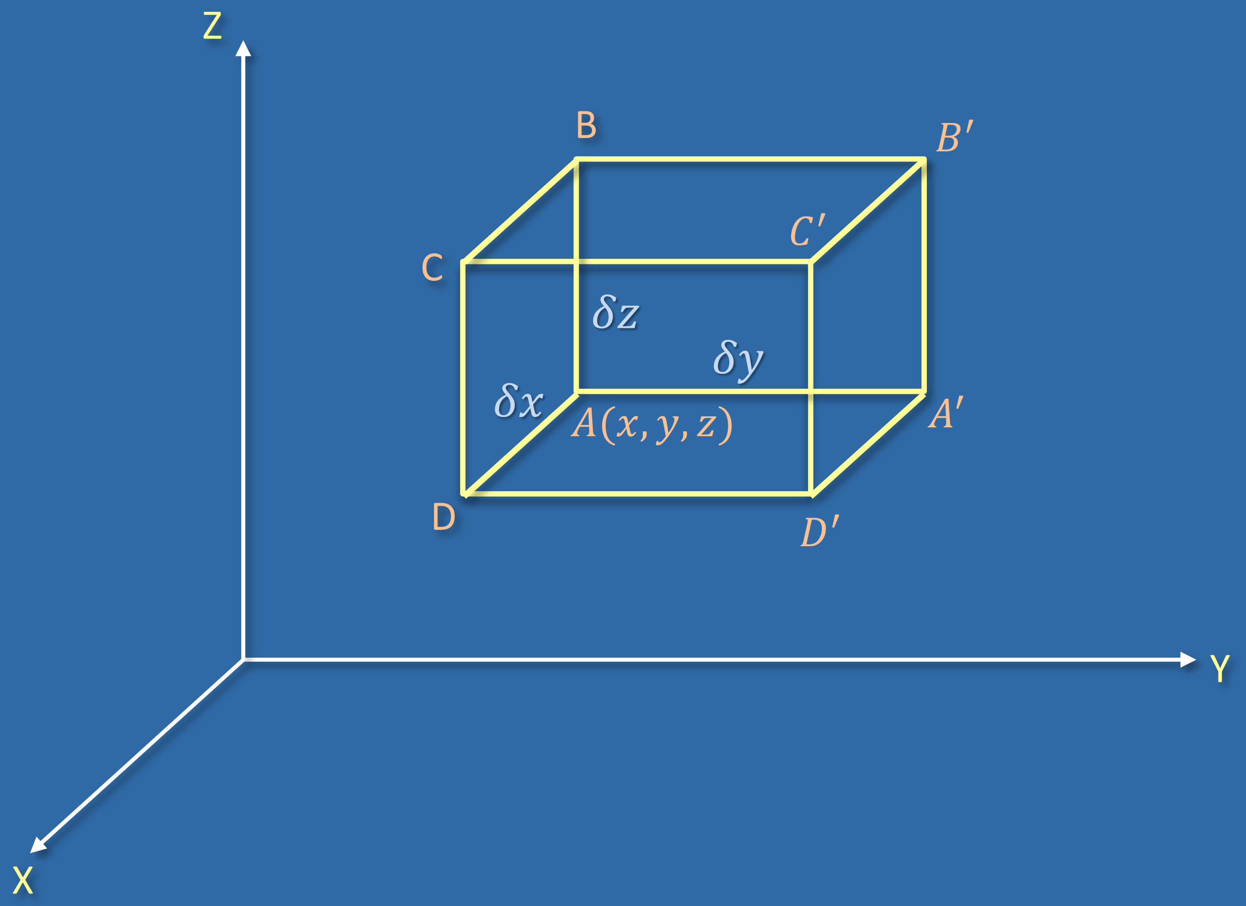
(Cartesian Form)


Z
Y
X


Z
A(x, y, z)
Y
X

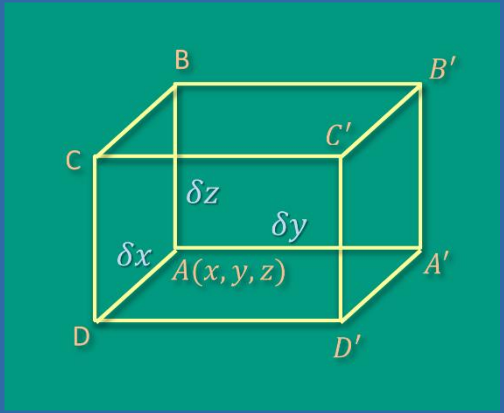

Z
w
v
u
A(x, y, z)
Y
X


Z
B
B'
C'
C
δz
δy
δx
$A(x, y, z)$
A'
D
D'
Y
X


Z
B
B'
C'
C
δz
δy
δx
$A(x, y, z)$
A'
D
D'
Y
X

Let,

- Fluid particle at $A(x, y, z)$.
- ρ be the density and p be the pressure.
- u, v, w be the velocity components at A parallel to the rectangular coordinate axes.
- (X, Y, Z) be the components of external force per unit mass at time t.



Let us construct a small parallelepiped with edges $\delta x, \delta y, \delta z$ of lengths parallel to their respective coordinate axes, having A at one of the angular points as shown in the figure.

$$\text{Force} = \text{Pressure} \times \text{Area}$$

Force due to pressure on the face $ABCD = p\delta y\delta z = f(x, y, z)$ (1)

Force due to pressure on the face $A'B'C'D' = f(x + \delta x, y, z)$

$$= f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + \cdots$$

$$= f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) \qquad (2)$$

Resultant force due to pressure along $x - axis$ is

$$(2)\text{-}(1) = -\delta x \frac{\partial}{\partial x} f(x, y, z) = -\frac{\partial p}{\partial x} \delta x \delta y \delta z \qquad (3)$$

- Du/Dt is the total acceleration of the element in x-direction.
- The mass of the element is $\rho \delta x \delta y \delta z$.
- The external force on the element in x-direction is $X \rho \delta x \delta y \delta z$.

By *Newton's second law of motion*, the equation of motion in x-direction is

Mass × (acceleration in x-direction) = Sum of the components of external forces in x-direction

$$i.e.\,\rho\delta x\delta y\delta z\frac{Du}{Dt} = X\rho\delta x\delta y\delta z - \frac{\partial p}{\partial x}\delta x\delta y\delta z$$

or,

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (4)$$

Similarly, the equations of motion in *y* and *z*-directions are, respectively

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (5)$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (6)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Rewriting (4), (5) and (6) the so-called Euler's dynamical equations of motion in cartesian coordinates are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{8}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{9}$$

Multiplying equations (4) by $\hat{\imath}$, (5) by $\hat{\jmath}$ and (6) by $\hat{k}$ we get,

$$\frac{Du}{Dt}\hat{\imath} = X\hat{\imath} - \frac{1}{\rho}\frac{\partial p}{\partial x}\hat{\imath}$$

$$\frac{Dv}{Dt}\hat{\jmath} = Y\hat{\jmath} - \frac{1}{\rho}\frac{\partial p}{\partial y}\hat{\jmath}$$

$$\frac{Dw}{Dt}\hat{k} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k}$$

Adding all the three equations we get,

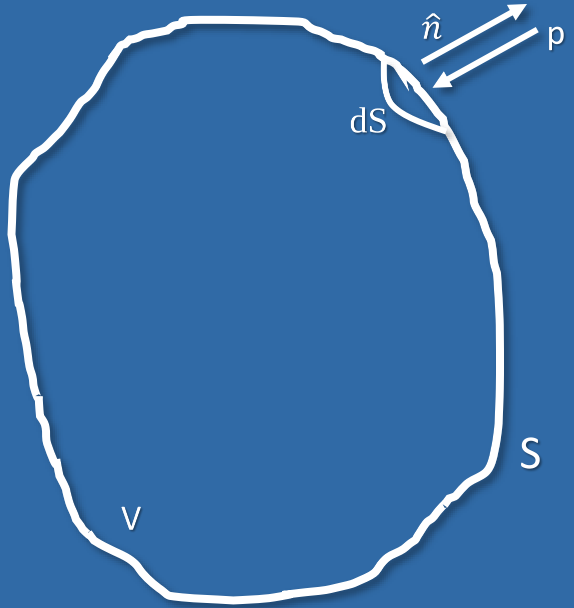
$$\frac{D}{Dt}\left(u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right) = (X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}) - \frac{1}{\rho}\left[\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right]p$$

$$=> \frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

which is *Euler's equation of motion*.

Euler's Equation of Motion

(Vector Method)



$\hat{n}$
p
dS
S
V

By *Newton's second law of motion,*

The total force acting on the mass of the fluid = rate of change of momentum

Total force = surface force + body force

Momentum = mass × velocity = density × volume ×velocity

$$\text{Mass} = \text{density} \times \text{volume} = \rho dV$$

$$\text{Momentum} = \text{velocity} \times \text{mass} = \vec{q}\rho dV$$

$$\text{Total momentum, } \vec{M} = \int_V \vec{q}\rho dV$$

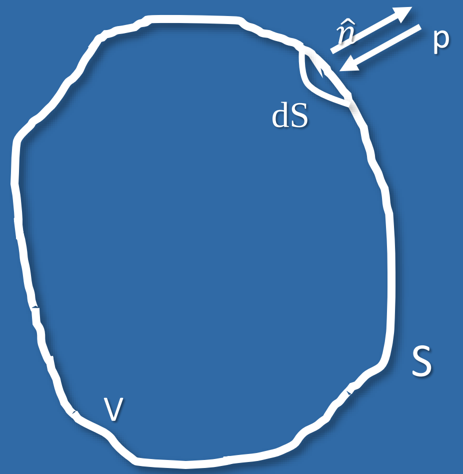
$$\text{Rate of change of momentum} = \frac{d}{dt}(\vec{M})$$

$$\frac{d\vec{M}}{dt} = \frac{d}{dt}\int_V \vec{q}\rho dV = \int_V \frac{d\vec{q}}{dt}\rho dV + \int_V \vec{q}\frac{d}{dt}(\rho dV)$$

$$\frac{d\vec{M}}{dt} = \int_V \frac{d\vec{q}}{dt}\rho dV$$

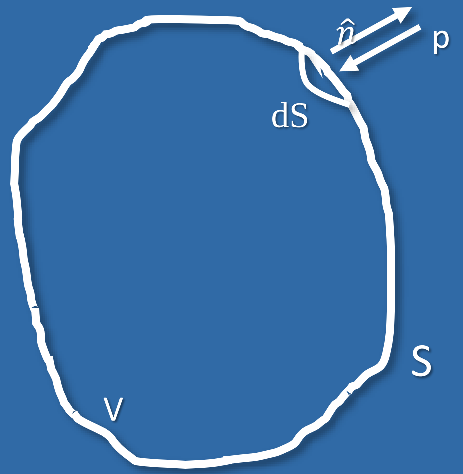
[Since mass is constant]

Force = pressure × area



Surface force on dS $= pdS(-\hat{n})$

Force = pressure × area

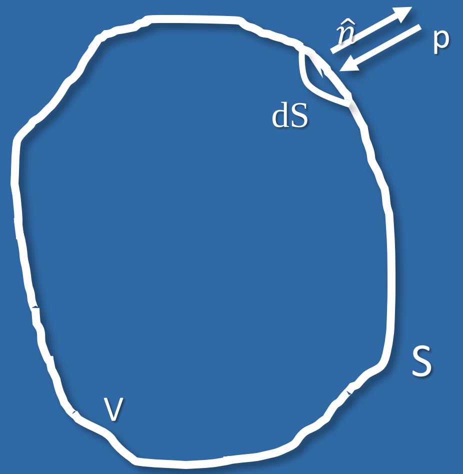


$$\text{Total surface force on S} = \int_S p\,dS(-\hat{n})$$

$$= -\int_S p\hat{n}\,dS$$

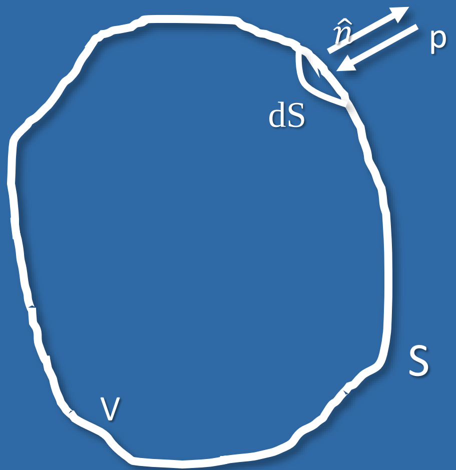
$$= -\int_V \nabla p\,dV$$

Force = pressure × area



Body force on dS $= \vec{F}\rho dV$

Force = pressure × area



Total body force on S $= \int_V \vec{F} \rho dV$

$\therefore$ Total force $= \int_V \vec{F} \rho dV - \int_V \nabla p dV$

From *Newton's second law of motion,*

Total force = Rate of change of momentum

$$\int_V \frac{d\vec{q}}{dt}\rho dV = \int_V \vec{F}\rho dV - \int_V \nabla p dV$$

$$=> \int_V \left[\frac{d\vec{q}}{dt}\rho - \vec{F}\rho + \nabla p\right] dV = 0$$

$$=> \frac{d\vec{q}}{dt}\rho - \vec{F}\rho + \nabla p = 0$$

$$=> \frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p$$

which is *Euler's equation of motion*.

Thank You