NASC for a Curve to be Geodesic

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MATHEMATICAL EXPLORATIONS

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NASC for a Curve to be Geodesic

A necessary and sufficient condition (NASC) for a curve $u = u(t)$, $v = v(t)$ on a surface $r = r(u, v)$ to be geodesic is that

$$V\left(\frac{\partial T}{\partial \dot{u}}\right) - U\left(\frac{\partial T}{\partial \dot{v}}\right) = 0$$

where

$$U \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}}$$

$$V \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}}$$

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Proof: Let A, B be **any two points** on a given surface $r = r(u, v)$.
Let us consider the arcs which join A and B by the equations of the form $u = u(t)$, $v = v(t)$ where $u(t)$ and $v(t)$ are functions of class 2.
Let us consider an arc α, $t = 0$ at A and $t = 1$ at B. Then length α is given by

$$S(\alpha) = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt \tag{1}$$

Let α be slightly deformed to obtain another curve α' , keeping the endpoints fixed at A and B. Then

$$u'(t) = u(t) + \epsilon g(t), \quad v'(t) = v(t) + \epsilon h(t)$$

where, ϵ is small and g and h are arbitrary functions of t of class 2 in $0 \leq t \leq 1$ and $g(0) = g(1) = 0$ and $h(0) = h(1) = 0$.

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Then from equation (1) we get,

$$S(\alpha') = \int_0^1 \sqrt{E\dot{u'}^2 + 2F\dot{u'}\dot{v'} + G\dot{v'}^2}\, dt \tag{2}$$

Let,

$$f = \sqrt{2T}$$

where,

$$T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)$$

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Then the difference in the **functionals** is given by

$$S(\alpha') - S(\alpha) = \int_0^1 [f(u+\epsilon g, v+\epsilon h, \dot{u}+\epsilon g, \dot{v}+\epsilon h) - f(u,v,\dot{u},\dot{v})]\, dt$$

Using **Taylor's theorem**,

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 \left[g\frac{\partial f}{\partial u} + h\frac{\partial f}{\partial v} + \dot{g}\frac{\partial f}{\partial \dot{u}} + \dot{h}\frac{\partial f}{\partial \dot{v}} \right] dt + O(\epsilon^2)$$

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 \left[g\left(\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)\right) + h\left(\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)\right) \right] dt + O(\epsilon^2)$$

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Let,

$$L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right), \quad M = \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)$$

Then,

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 (gL + hM)dt + O(\epsilon^2)$$

Thus, by definition, $S(\alpha)$ will be **stationary** and α **a geodesic** **if and only if** $u(t)$ are such that

$$\int_0^1 (gL + hM)dt = 0 \tag{3}$$

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for all possible g and h which satisfy $g = h = 0$ when $t = 0$ and $t = 1$.

We have, if $g(t)$ is continuous for $0 < t < 1$ and if

$$\int_0^1 V(t)g(t)\,dt = 0$$

for all admissible functions $V(t)$ as defined above, then $g(t) = 0$.

Since E, F, G are assumed to be of class 1 and $u(t), v(t)$ of class 2 implies that the functions L and M are continuous.

Now taking $h = 0$ & g, L in place of V, g, we get $L = 0$.
Similarly, by taking h, M in place of V, g, we get $M = 0$.

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Thus, (3) is satisfied for all admissible functions g and h if and only if

$$L = M = 0 \tag{4}$$

Since equations $L = 0$ and $M = 0$ do not involve the points A and B explicitly and therefore these equations are the same for all geodesics on the surface.
Now since $f = \sqrt{2T}$, we have

$$\begin{aligned} L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) &= \frac{1}{2}(2T)^{-1/2}2\frac{\partial T}{\partial u} - \frac{d}{dt}\left[\frac{1}{2}(2T)^{-1/2}2\frac{\partial T}{\partial \dot{u}}\right] \\ &= \frac{1}{\sqrt{2T}}\left[\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)\right] + \frac{1}{(2T)^{3/2}}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}} \qquad (5) \\ M &= \frac{1}{\sqrt{2T}}\left[\frac{\partial T}{\partial v} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right)\right] + \frac{1}{(2T)^{3/2}}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}} \qquad (6) \end{aligned}$$

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Therefore, geodesic equations are given by $(L = 0, M = 0)$ i.e.,

$$U \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}} \qquad \cdots (A)$$

$$V \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}} \qquad \cdots (B)$$

where U and V denote L.H.S. members of equations (A) and (B). Eliminating $\frac{dT}{dt}$ between the two equations (A) and (B), we get

$$V\left(\frac{\partial T}{\partial \dot{u}}\right) - U\left(\frac{\partial T}{\partial \dot{v}}\right) = 0$$

which is the necessary and sufficient condition for a curve on a surface to be geodesic.

Canonical Geodesic Equations

We have,

$$f = \sqrt{2T}$$

where,

$$T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)$$

Also,

$$U \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}} \qquad \cdots (A)$$

$$V \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}} \qquad \cdots (B)$$

Canonical Geodesic Equations

Since the parameter t is arbitrary, therefore replacing t by s and denoting the differentiation with respect to s by prime, we get

$$\begin{aligned} 2T &= Eu'^2 + 2u'v' + Gv'^2 \\ \Rightarrow 2T &= \frac{E\,du^2 + 2F\,du\,dv + G\,dv^2}{ds^2} \\ \Rightarrow 2T &= \frac{ds^2}{ds^2} \\ \Rightarrow T &= \frac{1}{2} \\ \Rightarrow \frac{dT}{ds} &= 0 \end{aligned} \tag{7}$$

Canonical Geodesic Equations

In view of this, equations (A) and (B) take the form

$$U \equiv \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0 \quad \cdots (C)$$

$$V \equiv \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \quad \cdots (D)$$

These are known as **canonical equations for geodesics**.

Remark 1. In equations (C) and (D), the partial derivatives are calculated from the relation

$$2T = Eu'^2 + 2Fu'v' + Gv'^2$$

before substituting the values for u' and v'. Actually, T is not equal to $\frac{1}{2}$ identically, for all $u, v, \dot{u}, \dot{v}$, but only along the curve.

Canonical Geodesic Equations

Remark 2. The identity

$$u'U + v'V = \frac{dT}{ds} \tag{8}$$

reduces to

$$u'U + v'V = 0 \quad (\text{for } \frac{dT}{ds} = 0) \tag{9}$$

Therefore equations (C) and (D) are not independent.
Again, for non-parametric curves $u' \neq 0, v' \neq 0$, we have from

$$u'U + v'V = 0 \tag{10}$$

that if $U = 0$, then $V = 0$, and if $V = 0$, then $U = 0$. Hence, the conditions $U = 0$ and $V = 0$ are equivalent to each other, being sufficient for a geodesic.

Canonical Geodesic Equations

Now, for a parametric curve $u = \text{constant}$, we have $u' = 0 \Rightarrow V = 0$ and therefore $U = 0$ for all s, and hence the equation $U = 0$ is satisfied automatically.

Thus, $U = 0$ is the condition for a geodesic in this case.

Similarly, we can show that $V = 0$ is the sufficient condition for a curve $v = \text{constant}$ to be a geodesic.

Show that a necessary and sufficient condition for a curve $v =$ constant, to be geodesic on the general surface is

$$EE_2 + FE_1 - 2EF_1 = 0$$

Proof: On the curve $v =$ const., u may be taken as a parameter so that $v = c, u = t$ represent the parametric equations of the curve. Therefore,

$$\dot{u} = 1, \quad \dot{v} = 0 \tag{1}$$

Thus,

$$2T = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \tag{2}$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left(E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2\right) \tag{3}$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left(E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2\right) \tag{4}$$

Theorem

$$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v}, \quad \frac{\partial T}{\partial \dot{v}} = F\dot{u} + G\dot{v} \tag{5}$$

When $\dot{u} = 1$ and $\dot{v} = 0$, we have

$$\frac{\partial T}{\partial u} = \frac{1}{2}E_1, \quad \frac{\partial T}{\partial v} = \frac{1}{2}E_2, \quad \frac{\partial T}{\partial \dot{u}} = E, \quad \frac{\partial T}{\partial \dot{v}} = F \tag{6}$$

$$\begin{aligned} U &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{dE}{dt} - \frac{1}{2}E_1 \\ &= \left(\frac{\partial E}{\partial u}\frac{du}{dt} + \frac{\partial E}{\partial v}\frac{dv}{dt}\right) - \frac{1}{2}E_1 \\ &= E_1\dot{u} + E_2\dot{v} - \frac{1}{2}E_1 = E_1 \cdot 1 - \frac{1}{2}E_1 = \frac{1}{2}E_1 \end{aligned} \tag{7}$$

Theorem

$$
\begin{aligned}
V &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{dF}{dt} - \frac{1}{2}E_2 \\
&= F_1 - \frac{1}{2}E_2
\end{aligned} \tag{8}
$$

Thus the curve $v = c$ is a geodesic, i.e., if

$$
\begin{aligned}
& U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0 \\
\Rightarrow & \frac{1}{2}E_1 F - \left(F_1 - \frac{E_2}{2}\right)E = 0 \\
\Rightarrow & EE_2 + FE_1 - 2EF_1 = 0, \quad \text{when } v = c, \forall u \quad \blacksquare
\end{aligned}
$$

Examples

Q. Show that the curves $u + v = constant$ are geodesics on a surface with metric

$$(1+u^2)du^2 - 2uv\,du\,dv + (1+v^2)dv^2$$

Solution: The parametric equations of the given curve $u + v =$ constant can be taken as $u = t, v = c - t$ so that

$$\dot{u} = 1, \quad \dot{v} = -1 \tag{1}$$

$$\text{Here,} \quad E = (1+u^2), \quad F = -uv, \quad G = (1+v^2) \tag{2}$$

Now,

$$T = \frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2] \tag{3}$$

Examples

Substituting the values of E, F, G, we get:

$$T = \frac{1}{2}[(1+u^2)\dot{u}^2 - 2uv\dot{u}\dot{v} + (1+v^2)\dot{v}^2] \quad (4)$$

$$\frac{\partial T}{\partial u} = u\dot{u}^2 - v\dot{u}\dot{v} = t - (c-t)(-1) = c \quad (5)$$

$$\frac{\partial T}{\partial v} = -u\dot{v} + v\dot{v}^2 = t + c - t = c \quad (6)$$

$$\frac{\partial T}{\partial \dot{u}} = (1+u^2)\dot{u} - u\dot{v} = (1+t^2)(1) - t(c-t)(-1) = 1 + ct \quad (7)$$

$$\frac{\partial T}{\partial \dot{v}} = -u\dot{v} + (1+v^2)\dot{v} = -t(c-t) + [1+(c-t)^2](-1) = ct - 1 - c^2 \quad (8)$$

Examples

Now,

$$U \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(1 + ct) - c = 0 \tag{9}$$

$$V \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(ct - 1 - c^2) - c = 0 \tag{10}$$

Hence, the relation

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0$$

for all values of t, proving that the given curve $u + v =$ constant is a geodesic.

Prove that the curves of the family $v^3/u^2 = constant$ are geodesics on a surface with metric

$$v^2du^2 - 2uv\,du\,dv + u^2dv^2 \quad (u > 0, v > 0).$$

Solution: The parametric equations of the given family of curves $v^2 = cu^2$ can be taken as $u = ct^3$, $v = ct^2$, where c is any constant. Then

$$\dot{u} = 3ct^2, \quad \dot{v} = 2ct \tag{1}$$

Also,

$$E = v^2, \quad F = -uv, \quad G = 2u^2.$$

Examples

Therefore,

$$T = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right) = \frac{1}{2}(v^2\dot{u}^2 - 2uv\dot{u}\dot{v} + 2u^2\dot{v}^2)$$

$$\frac{\partial T}{\partial u} = -u\dot{u}\dot{v} + 2u\dot{v}^2 = -(ct^2)(3ct^2)(2ct) + 2(ct^3)(2ct)^2 = -6c^3t^5 + 8c^3t^5 = 2c^3t^5$$

$$\frac{\partial T}{\partial v} = v\dot{u}^2 - u\dot{u}\dot{v} = 3c^3t^6$$

$$\frac{\partial T}{\partial \dot{u}} = v^2\dot{u} - uv\dot{v} = c^3t^6$$

$$\frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + 2u^2\dot{v} = c^3t^7$$

Examples

Now,

$$U \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(c^4t^6) - 2c^3t^5 = 4c^3t^5$$

$$V \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(c^3t^7) - 3c^3t^6 = 4c^3t^6$$

$$\therefore \quad V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = (4c^3t^6)(c^3t^6) - (4c^3t^5)(c^2t^7) = 0$$

Thus, the given family of curves $v^3/u^2 = c$ is a geodesic for all values of c.

THANK YOU

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