Relativity & Cosmology 1

Presented by

MATHEMATICAL EXPLORATIONS

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Statement:

The Principle of Equivalence is a fundamental concept in Einstein's theory of General Relativity. It states that the effects of gravity and acceleration are indistinguishable in a small, localized region of space. This means that an observer in a closed system (like an elevator or spaceship) cannot tell whether the force they experience is due to gravity or uniform acceleration.

In simple terms: Being in a gravitational field is equivalent to being in an accelerated frame of reference.

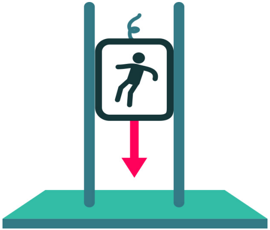

Figure 1: Inside elevator

Types of Equivalence Principles

There are two main versions of this principle:

Weak Equivalence Principle (WEP)

Also known as the Galilean Equivalence Principle, it states that the motion of an object in a gravitational field is independent of its mass or composition.
Example: A feather and a hammer, when dropped in a vacuum (like on the Moon, as shown by Apollo 15 astronauts), fall at the same rate.

Einstein's Strong Equivalence Principle (SEP)

It extends the WEP by stating that locally, the effects of gravity are indistinguishable from acceleration. This means that being in a gravitational field is equivalent to being in an accelerated frame of reference.
Example: A person in free fall can't distinguish between gravity and acceleration.

Mach's Principle

Mach's Principle is a philosophical and physical idea proposed by Ernst Mach (1838–1916), which suggests that the **inertia** of an object arises from its interaction with the total mass of the universe.

Core Idea

- The inertia (resistance to acceleration) of an object is not an inherent property but is determined by the presence and distribution of all other masses in the universe.
- In simpler terms, if the universe were empty, there would be no inertia—an isolated object wouldn't "resist" acceleration because there would be nothing else to define motion.

Explanation of Mach's Principle

1. **Inertia is a Collective Effect**
   - When you push an object, its resistance to motion (inertia) is due to the **gravitational influence** of distant stars and galaxies.
2. **Rotating Bucket Experiment**
   - If you spin a bucket of water, the water's surface **curves** (centrifugal effect).
   - According to Mach's Principle, the reason this happens is that the entire universe provides a **reference frame** for rotation.
   - If the stars and galaxies didn't exist, would the water still form a curved surface? Mach's Principle suggests it wouldn't because rotation is only meaningful relative to the **mass** of the universe.
3. **No Absolute Space**
   - Unlike Newton's idea of **absolute space**, Mach's Principle suggests that motion is always defined relative to other objects.

Principle of Covariance

The Principle of Covariance states that the fundamental laws of physics must have the same form in all coordinate systems. This principle ensures that physical laws are **independent** of the observer's choice of reference frame.

Understanding the Principle:

1. **General Covariance (Einstein's Definition)**
   - The equations describing physical laws should **remain valid** under any coordinate transformation (including transformations between accelerating frames).
   - This is a key idea in General Relativity, where Einstein formulated gravitational laws using tensor equations, ensuring that they **hold true** in any reference frame.
2. **Special Covariance (Restricted Case)**
   - Some physical laws may be valid only in specific types of coordinates, such as inertial frames (like in Special Relativity).

Geodesic Principle

The Geodesic Principle is a fundamental concept in General Relativity that describes how **objects move** in curved spacetime. It states that:

> An object moving under gravity alone (without any external force) follows the shortest or straightest possible path in curved spacetime, called a **geodesic**.

A geodesic is like a **"straight line"** in curved spacetime, similar to how a great circle (like the equator) is the shortest path between two points on a sphere.

- In **flat space** (no gravity), a geodesic is just a **straight line**.
- In **curved space** (with gravity), a geodesic appears **curved** because space and time themselves are curved by mass and energy.

Geodesic Principle

Geodesic Motion in General Relativity

- In Einstein's theory, gravity is not a force but a result of spacetime curvature.
- Objects move along geodesics naturally, just like a ball rolling on a curved surface.
- This explains why planets orbit the Sun. They are not "pulled" by a force but instead follow geodesics in the curved spacetime created by the Sun's mass.
  Example: Imagine you are inside an elevator in deep space, far from any gravity. If the elevator is turned off, you float weightlessly. Now, imagine the elevator is falling freely under gravity near Earth. Inside, you again feel weightless, because both you and the elevator are following the same geodesic. This is why astronauts in orbit feel weightless—they are in free fall along a geodesic of curved spacetime.

Newtonian Approximation of Relativistic Equations of Motion

When studying motion, we usually start with Newton's laws of motion, which describe how objects move under the influence of forces. However, when an object moves very fast—close to the speed of light (c), Newtonian physics no longer provides accurate results. Instead, we use Einstein's Special Theory of Relativity, which gives the correct equations for motion at high speeds.

However, if the speed of the object is much slower than the speed of light $(v \ll c)$, then relativistic motion should simplify to Newtonian motion. This process is called the **Newtonian approximation** of relativistic equations of motion.

Newton's Law of Momentum:

In Newtonian mechanics, momentum is given by:

$$p = mv \tag{1}$$

Newtonian Approximation of Relativistic Equations of Motion

where:

- p = momentum,
- m = mass,
- v = velocity of the object.

Relativistic Momentum:

When an object moves at speeds close to c, its mass appears to increase due to relativistic effects. The correct equation for momentum is:

$$p = \gamma m v \tag{2}$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3}$$

Newtonian Approximation of Relativistic Equations of Motion

Newtonian Approximation: If $v \ll c$, we use a Taylor expansion for γ to approximate:

$$\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \tag{4}$$

Substituting this into the relativistic momentum equation:

$$p \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) mv \tag{5}$$

For very small v/c, we can ignore $\frac{1}{2}\frac{v^2}{c^2}$, so:

$$p \approx mv \tag{6}$$

This is exactly the Newtonian momentum equation, meaning Newtonian mechanics is a good approximation at low speeds.

Newtonian Approximation of Relativistic Equations of Motion

Relativistic Energy Equation:

Einstein's famous energy equation is:

$$E = \gamma mc^2 \tag{7}$$

Expanding γ for small speeds:

$$E \approx mc^2 + \frac{1}{2}mv^2 \tag{8}$$

- The first term mc^2 is the rest energy (exists even when the object is not moving).
- The second term $\frac{1}{2}mv^2$ is the familiar Newtonian kinetic energy.

At low speeds, Einstein's energy equation reduces to the Newtonian K.E.

Newtonian Approximation of Relativistic Equations of Motion

Newton's Second Law:

$$F = m\frac{dv}{dt} \tag{9}$$

This states that force is equal to mass times acceleration.

Relativistic Force Equation:

In relativity, force is given by:

$$F = \frac{d}{dt}(\gamma m v) \tag{10}$$

For slow speeds $(v \ll c)$, since $\gamma \approx 1$, this simplifies to:

$$F \approx m\frac{dv}{dt} \tag{11}$$

Newtonian Approximation of Relativistic Equations of Motion

Newtonian Motion as a Limit of Relativity

- Relativistic momentum reduces to $p = mv$ when $v \ll c$.
- Relativistic energy reduces to $E = \frac{1}{2}mv^2$ plus rest energy.
- Relativistic force equation simplifies to Newton's second law.
- Newtonian mechanics is just a low-speed approximation of relativity.

Newtonian Eqns of Motion as an Approx. of Geodesic Eqns

Geodesic equations are reducible to Newtonian equations of motion in case of weak static field.

Proof: Let us consider the motion of a test particle in case a weak static field. The motion of a test particle is governed by geodesic equations as given below:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{1}$$

Since the field is static, *i.e.*, it does not change with time. Hence, velocity components can be taken as:

$$\frac{dx^1}{ds} = \frac{dx^2}{ds} = \frac{dx^3}{ds} = 0; \quad \text{and} \quad \frac{dx^4}{ds} = 1 \tag{2}$$

Newtonian Equation of Motion - Geodesic Equations

Galilean coordinates are:

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct \tag{3}$$

A weak static field is characterized by taking:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{such that} \quad g_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu \tag{4}$$

where,

- $\eta_{\mu\nu}$ is the **Minkowski metric** (Galilean values in a weak field limit).
- $h_{\mu\nu}$ represents **small perturbations** in the metric caused by gravity.

Newtonian Equation of Motion - Geodesic Equations

Nature of $h_{\mu\nu}$

- The function $h_{\mu\nu}$ depends only on spatial coordinates (x, y, z) and is independent of time, ensuring that the gravitational field is static.
- The deviation from unity is represented by $h_{\mu\nu}$ and it is assumed to be small so that terms involving higher powers of $h_{\mu\nu}$ (like $h^2_{\mu\nu}$) can be neglected.

Metric Components in a Weak Static Field

The Minokswki metric for this weak field approximation is

$$\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{44} = -1, \quad \eta_{\mu\nu} = 0 = g_{\mu\nu} \quad (\mu \neq \nu) \tag{5}$$

- The spatial components η_{11}, η_{22}, η_{33} correspond to the three spatial dimensions.
- η_{44} corresponds to the time component.

Newtonian Equation of Motion - Geodesic Equations

The metric equation in general relativity is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{6}$$

where:

- ds is the infinitesimal **proper time** interval (or spacetime interval),
- $g_{\mu\nu}$ is the **metric tensor**,
- dx^μ and dx^ν are infinitesimal **coordinate displacements**.

Dividing both sides by ds^2, we obtain:

$$1 = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \tag{7}$$

This equation represents the normalization condition for a timelike geodesic.

Newtonian Equation of Motion - Geodesic Equations

In the weak static field approximation, only the time component (g_{44}) significantly contributes, while spatial components remain **small or negligible**, so

$$1 = g_{44} \frac{dx^4}{ds} \frac{dx^4}{ds} \tag{8}$$

Since x^4 corresponds to time, we substitute $x^4 = ct$ (where c is the speed of light and t is coordinate time), so

$$dx^4 = cdt. \tag{9}$$

Thus,

$$1 = g_{44} \left(c\frac{dt}{ds} \right) \left(c\frac{dt}{ds} \right) \tag{10}$$

Newtonian Equation of Motion - Geodesic Equations

From weak-field gravity, we assume:

$$g_{44} = 1 + h_{44}, \tag{11}$$

where h_{44} represents **small perturbations** in the gravitational field. Substituting this into the equation (10):

$$\begin{aligned} 1 &= (1 + h_{44})c^2 \frac{dt}{ds} \cdot \frac{dt}{ds} \\ => ds^2 &= (1 + h_{44})c^2 dt^2 \end{aligned} \tag{12}$$

Taking the first approximation, we assume that the perturbation h_{44} is small and can be neglected in this initial step. This simplifies the equation to:

$$\begin{aligned} ds^2 &= c^2 dt^2 \\ => ds &= cdt \end{aligned} \tag{13}$$

Newtonian Equation of Motion - Geodesic Equations

From equation (2), since we are considering the weak-field case where only the time component dominates, we take $\mu = 4$ and $\nu = 4$, reducing the equation to:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{44}\frac{dx^4}{ds}\frac{dx^4}{ds} = 0 \tag{14}$$

We have,

$$x^4 = ct \Rightarrow \frac{dx^4}{ds} = c\frac{dt}{ds} \tag{15}$$

Again,

$$ds = cdt \Rightarrow \frac{dt}{ds} = \frac{1}{c} \tag{16}$$

Newtonian Equation of Motion - Geodesic Equations

Therefore,

$$\frac{dx^4}{ds} = 1 \tag{17}$$

Substituting this into the geodesic equation:

$$\begin{aligned} &\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{44}(1)^2 = 0 \Rightarrow \frac{d^2x^\alpha}{ds^2} = -\Gamma^\alpha_{44} \\ &\Rightarrow -\Gamma^\alpha_{44} = \frac{d}{ds}\left(\frac{dx^\alpha}{ds}\right) = \frac{d}{cdt}\left(\frac{dx^\alpha}{cdt}\right) = \frac{1}{c^2}\frac{d^2x^\alpha}{dt^2} \\ &\Rightarrow \frac{d^2x^\alpha}{dt^2} = -c^2\Gamma^\alpha_{44} \end{aligned} \tag{18}$$

Newtonian Equation of Motion - Geodesic Equations

Since the field is **static**,

$$\Gamma^4_{44} = 0 \tag{19}$$

Hence,

$$\frac{d^2x^\alpha}{dt^2} = -c^2\Gamma^\alpha_{44}, \quad where \quad \alpha = 1, 2, 3 \tag{20}$$

Expanding the **Christoffel symbols**,

$$\Gamma^\alpha_{44} = g^{\alpha\beta}\Gamma_{44,\beta} = g^{\alpha\alpha}\Gamma_{44,\alpha} = g^{\alpha\alpha}\frac{1}{2}\left(2\frac{\partial g_{\alpha 4}}{\partial x^\alpha} - \frac{\partial g_{44}}{\partial x^\alpha}\right)$$
$$= \frac{1}{2g_{\alpha\alpha}}\left[-\frac{\partial}{\partial x^\alpha}(1 + h_{44})\right] = \frac{1}{2}(-1 + h_{\alpha\alpha})^{-1}\left(-\frac{\partial h_{44}}{\partial x^\alpha}\right)$$

Newtonian Equation of Motion - Geodesic Equations

$$
\begin{aligned}
&= \frac{1}{2}\frac{1}{(-1+h_{\alpha\alpha})}\left(-\frac{\partial h_{44}}{\partial x^{\alpha}}\right) \\
&= \frac{1}{2}\frac{1}{(-1)(1-h_{\alpha\alpha})}\left(-\frac{\partial h_{44}}{\partial x^{\alpha}}\right) \\
&= \frac{1}{2}(1-h_{\alpha\alpha})^{-1}\left(\frac{\partial h_{44}}{\partial x^{\alpha}}\right) \\
&= \frac{1}{2}(1+h_{\alpha\alpha})\left(\frac{\partial h_{44}}{\partial x^{\alpha}}\right) \qquad [\because\ h_{\alpha\alpha}\ is\ small, 1+h_{\alpha\alpha}\approx 1] \\
&= \frac{1}{2}\left(\frac{\partial h_{44}}{\partial x^{\alpha}}\right)
\end{aligned}
$$

Newtonian Equation of Motion - Geodesic Equations

Thus, equation (20) becomes

$$\frac{d^2x^\alpha}{dt^2} = -\frac{c^2}{2}\frac{\partial h_{44}}{\partial x^\alpha}. \tag{21}$$

The **Newtonian equation of motion** is given by,

$$\frac{d^2x^\alpha}{dt^2} = -\frac{\partial \psi}{\partial x^\alpha}, \tag{22}$$

where ψ **is the potential function**.
Equating equations (21) and (22) we get,

$$-\frac{c^2}{2}\frac{\partial h_{44}}{\partial x^\alpha} = -\frac{\partial \psi}{\partial x^\alpha}. \tag{23}$$

Integrating,

$$\int \frac{\partial h_{44}}{\partial x^\alpha} dx^\alpha = \frac{2}{c^2} \int \frac{\partial \psi}{\partial x^\alpha} dx^\alpha$$
$$\Rightarrow \int dh_{44} = \frac{2}{c^2} \int d\psi$$
$$\Rightarrow h_{44} = \frac{2\psi}{c^2} + \text{const}$$
$$\Rightarrow 1 + h_{44} = 1 + \frac{2\psi}{c^2} + \text{const}$$
$$\Rightarrow g_{44} = \frac{2\psi}{c^2} + k$$

Newtonian Equation of Motion - Geodesic Equations

Choosing ψ such that when $g_{44} = 1$, $\psi = 0$ so that $k = 1$. Then

$$g_{44} = 1 + \frac{2\psi}{c^2}$$

Hence, geodesic equations are reducible to Newtonian equations of motion in case of weak static field if $g_{44} = 1 + \frac{2\psi}{c^2}$.

THANK YOU

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