Poisson's Equation as an Approximation of Field Equations

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MATHEMATICAL EXPLORATIONS

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Poisson's Equation as an Approximation of Field Equations

To show that (Einstein's) field equations reduce in linear approximation to Newtonian equations (Poisson's equations)

$$\nabla^2 \psi = 4\pi\rho$$

Proof: Let us Consider the motion of a test particle in a weak static field. A weak static field is characterized by taking:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

where $\eta_{\mu\nu}$ is a metric tensor for Galilean line element and $h_{\mu\nu}$ is a function of x, y, z. The deviation of the metric from unity is represented through $h_{\mu\nu}$. The quantities $h_{\mu\nu}$ are taken to be so small that the powers of $h_{\mu\nu}$ higher than the first are neglected. Here we have:

$$\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{44} = -1, \quad \eta_{\mu\nu} = 0 = g_{\mu\nu} \quad \text{for } \mu \neq \nu \tag{2}$$

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Since the field is static, *i.e.*, it does not change with time and hence, velocity components can be taken as:

$$\frac{dx^1}{ds} = \frac{dx^2}{ds} = \frac{dx^3}{ds} = 0; \quad \text{and} \quad \frac{dx^4}{ds} = 1 \tag{3}$$

Galilean coordinates are:

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct \tag{4}$$

The geodesic equations are reduced to Newtonian equations of motion if

$$g_{44} = 1 + \frac{2\psi}{c^2}$$

Let, $c = 1$ then

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$$g_{44} = 1 + 2\psi \tag{5}$$

All the components of the energy tensor will be approximately equal to zero separately except

$$T_{44} = \rho$$

so that

$$\begin{aligned} T = g^{\mu\nu}T_{\mu\nu} &= g^{44}T_{44} \\ &= (1 + h_{44})^{-1}\rho \\ &= (1 - h_{44} + ...)\rho \\ &= \rho \end{aligned}$$

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$$\therefore T_{44} = \rho, \quad T = \rho \tag{6}$$

Field equations in general theory of relativity are given by,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{7}$$

By contracting both sides of the Einstein field equations with $g^{\mu\nu}$, we get

$$\begin{aligned} g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} &= -8\pi g^{\mu\nu} T_{\mu\nu} \\ \Rightarrow R - \frac{4}{2} R &= -8\pi T \\ \Rightarrow R &= 8\pi T \end{aligned} \tag{8}$$

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Then from equation (7) we get,

$$R_{\mu\nu} - \frac{1}{2}8\pi T g_{\mu\nu} = -8\pi T_{\mu\nu}$$
$$\Rightarrow R_{\mu\nu} = -8\pi\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) \quad (9)$$

Therefore,

$$\begin{aligned} R_{44} &= -8\pi\left(T_{44} - \frac{1}{2}T g_{44}\right) \\ &= -8\pi\left(\rho - \frac{1}{2}\rho.1\right) \quad [\because g_{44} \approx 1] \\ &= -4\pi\rho \end{aligned} \quad (10)$$

Poisson's Equation as an Approximation of Field Equations

The Riemann curvature tensor is defined as

$$
\begin{aligned}
R^a_{\mu\nu\sigma} &= -\frac{\partial \Gamma^a_{\mu\nu}}{\partial x^\sigma} + \frac{\partial \Gamma^a_{\mu\sigma}}{\partial x^\nu} + \Gamma^b_{\mu\nu}\Gamma^a_{b\sigma} - \Gamma^b_{\mu\sigma}\Gamma^a_{b\nu} \\
\therefore R_{44} = R^a_{44a} &= -\frac{\partial \Gamma^a_{44}}{\partial x^a} + \frac{\partial \Gamma^a_{a4}}{\partial x^4} + \Gamma^b_{44}\Gamma^a_{ba} - \Gamma^b_{4a}\Gamma^a_{b4} \qquad (11)
\end{aligned}
$$

Using first order approximation we get,

$$
R_{44} = -\frac{\partial \Gamma^a_{44}}{\partial x^a} + \frac{\partial \Gamma^a_{a4}}{\partial x^4} \qquad (12)
$$

Since in a static approximation (no explicit dependence on x^4, which represents time), the term $\frac{\partial \Gamma^a_{a4}}{\partial x^4}$ can be neglected, therefore

Newtonian Approximation of Relativistic Equations of Motion

$$R_{44} = -\frac{\partial \Gamma^a_{44}}{\partial x^a}$$
$$\Rightarrow \frac{\partial \Gamma^a_{44}}{\partial x^a} = 4\pi\rho \qquad (13)$$

Since the system is static, all components of metric tensor $g_{\mu\nu}$ are independent of time x^4.

$$\frac{\partial g_{\mu\nu}}{\partial x^4} = 0 \quad \forall\ \mu \text{ and } \nu$$
$$\therefore \quad \frac{\partial \Gamma^4_{44}}{\partial x^4} = 0 \qquad (14)$$

Newtonian Approximation of Relativistic Equations of Motion

Hence,

$$\frac{\partial \Gamma^a_{44}}{\partial x^a} = 4\pi\rho, \quad a = 1, 2, 3 \tag{15}$$

If $a = 1, 2, 3$, then

$$\begin{aligned}
\Gamma^a_{44} = g^{ab}\Gamma_{44,b} = g^{aa}\Gamma_{44,a} &= g^{aa}\frac{1}{2}\left(\frac{\partial g_{4a}}{\partial x^4} + \frac{\partial g_{4a}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^a}\right) \\
&= \frac{1}{-1 + h_{aa}}\frac{1}{2}\left(-\frac{\partial g_{44}}{\partial x^a}\right) \\
&= (1 - h_{aa})^{-1}\frac{1}{2}\frac{\partial g_{44}}{\partial x^a} \\
&= (1 + h_{aa} + \dots)\frac{1}{2}\frac{\partial g_{44}}{\partial x^a} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^a}
\end{aligned} \tag{16}$$

Newtonian Approximation of Relativistic Equations of Motion

Now equation (15) reduces to

$$\frac{\partial}{\partial x^a}\left(\frac{1}{2}\frac{\partial g_{44}}{\partial x^a}\right) = 4\pi\rho$$
$$\Rightarrow \sum_{a=1}^{3} \frac{\partial^2 g_{44}}{\partial x^a \partial x^a} = 8\pi\rho \tag{17}$$

By definition, the Laplacian ∇^2 in three-dimensional Cartesian coordinates is:

$$\nabla^2 g_{44} = \sum_{a=1}^{3} \frac{\partial^2 g_{44}}{\partial x^a \partial x^a} \tag{18}$$

Newtonian Approximation of Relativistic Equations of Motion

Thus from equations (17) and (18) we get,

$$\begin{aligned} \nabla^2 g_{44} &= 8\pi\rho \\ \Rightarrow \nabla^2(1+2\psi) &= 8\pi\rho \\ \Rightarrow \nabla^2\psi &= 4\pi\rho \end{aligned} \tag{19}$$

which is Poisson's equation.

THANK YOU

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