Energy-Momentum Tensor

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Energy-Momentum Tensor

Energy Momentum Tensor:

In General Relativity, **gravity** is not a force but a curvature of space-time caused by mass and energy. To describe how energy and momentum influence this curvature, Einstein introduced the **Energy-Momentum Tensor** (also called the **Stress-Energy Tensor**). The energy-momentum tensor is denoted by $T_{\mu\nu}$.

- This tensor is a mathematical object that encodes how energy, momentum, and stress (pressure and shear forces) are distributed in space-time.
- It plays a key role in Einstein's Field Equations, which determine how space-time bends in response to matter and energy.

Energy-Momentum Tensor

What is the Energy-Momentum Tensor?

- Imagine you have a fabric (representing space-time), and you place a **heavy object** on it (representing mass/energy). The fabric bends due to the weight of the object. The more massive or energetic the object, the **greater** the curvature.
- But how do we mathematically describe how much bending occurs? This is where the energy-momentum tensor comes in—it tells us how energy, momentum, and pressure are distributed in space-time, which then determines how **space-time curves**.

Components of the Energy-Momentum Tensor

$T_{\mu\nu}$ is a 4×4 matrix (tensor) that describes energy, momentum, and stress in four-dimensional space-time with the following structure:

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

- T_{00} (**Energy Density**): Represents the total energy (including mass and kinetic energy) in a given region.
- T_{0i} and T_{i0} (**Energy Flux / Momentum Density**): These components describe how energy moves through space (like heat flow or radiation).
- T_{ij} ((**Stress: Pressure and Shear Forces**): These terms describe internal forces in a system, like pressure in a gas or tension in a solid.

Why is the Energy-Momentum Tensor Important?

- **It describes all types of energy and momentum in space-time**: From ordinary matter (stars, planets) to radiation and dark energy, everything influencing gravity is contained in $T_{\mu\nu}$.
- **It makes General Relativity work in real-world physics**: Without the energy-momentum tensor, Einstein's equations would not explain how the universe behaves.
- **It applies to different fields of physics**: It is used in cosmology (universe expansion), astrophysics (black holes, neutron stars), and quantum field theory (particles and fields).

General Form of the Energy-Momentum Tensor

The general form is:

$$T^{\mu\nu} = \frac{1}{c^2}(\rho c^2 + p)U^\mu U^\nu + pg^{\mu\nu}$$

where:

- ρ = energy density of the fluid,
- p = pressure,
- U^μ = four-velocity of the fluid,
- $g^{\mu\nu}$ = metric tensor of space-time,
- c = speed of light.

This form is often used for a perfect fluid, such as a star or a gas cloud.

General Form of the Energy-Momentum Tensor

Using the metric tensor $g_{\mu\nu}$, we lower the indices:

$$
\begin{aligned}
T_{\mu\nu} &= g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta} \\
\Rightarrow T_{\mu\nu} &= g_{\mu\alpha} g_{\nu\beta} \left[\frac{1}{c^2} (\rho c^2 + p) U^\alpha U^\beta + p g^{\alpha\beta} \right] \\
\Rightarrow T_{\mu\nu} &= \frac{1}{c^2} (\rho c^2 + p) g_{\mu\alpha} g_{\nu\beta} U^\alpha U^\beta + p g_{\mu\alpha} g_{\nu\beta} g^{\alpha\beta} \\
\Rightarrow T_{\mu\nu} &= \frac{1}{c^2} (\rho c^2 + p) g_{\mu\alpha} U^\alpha g_{\nu\beta} U^\beta + p g_{\mu\alpha} g_{\nu\beta} g^{\alpha\beta} \\
\Rightarrow T_{\mu\nu} &= \frac{1}{c^2} (\rho c^2 + p) U_\mu U_\nu + p g_{\mu\nu}
\end{aligned}
$$

Forms of the Energy-Momentum Tensor

Energy-Momentum Tensor for Perfect Fluid

The energy-momentum tensor for a perfect fluid (without viscosity or heat conduction) is

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}$$

Energy-Momentum Tensor for Electromagnetic Field

For an electromagnetic field, the energy-momentum tensor is

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F_\alpha{}^\nu - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

- $F^{\mu\nu}$ = electromagnetic field tensor,
- μ_0 = permeability of free space.

Forms of the Energy-Momentum Tensor

Energy-Momentum Tensor for Vacuum Energy (Dark Energy & Cosmological Constant)

For dark energy or vacuum energy (cosmological constant Λ) energy-momentum tensor is given by

$$T^{\mu\nu} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}$$

This represents the energy associated with empty space, driving the accelerated expansion of the universe.

Energy-Momentum Tensor in Galilean Coordinates

In Newtonian mechanics, momentum is given by:

$$p^i = mv^i \tag{1}$$

where

$$v^i = \frac{dx^i}{dt}. \tag{2}$$

When dealing with a continuous medium (instead of a single particle), we replace the mass m with mass density ρ_0:

$$T^{ij} = \rho_0 v^i v^j. \tag{3}$$

Energy-Momentum Tensor in Galilean Coordinates

Since we are considering a gravitational system where motion is described in terms of proper time s instead of coordinate time t, we generalize the velocity:

$$v^i = \frac{dx^i}{ds}. \tag{4}$$

Thus, we arrive at the equation:

$$T^{ij} = \rho_0 \frac{dx^i}{ds}\frac{dx^j}{ds}. \tag{5}$$

In the Galilean coordinate system, we have

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2, \quad \text{when } c = 1.$$

Energy-Momentum Tensor in Galilean Coordinates

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = -\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 + 1$$
$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = 1 - v^2 \quad \left[\text{where} \quad v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, c = 1\right] \tag{6}$$

If ρ be the coordinate density of matter and v the velocity relative to the Galilean coordinate system, then we have the relation

$$\rho = \frac{\rho_0}{1 - \frac{v^2}{c^2}} = \frac{\rho_0}{1 - v^2} \quad [\text{ If } c = 1, \text{ i.e., in a gravitational system.}]$$

$$\Rightarrow \rho(1 - v^2) = \rho_0$$
$$\Rightarrow \rho\left(\frac{ds}{dt}\right)^2 = \rho_0 \tag{7}$$

Using this in equation (5) we get,

$$T^{ij} = \rho_0 \frac{dx^i}{ds}\frac{dx^j}{ds}$$
$$\Rightarrow T^{ij} = \rho_0 \frac{dx^i}{dt}\frac{dt}{ds}\frac{dx^j}{dt}\frac{dt}{ds}$$
$$\Rightarrow T^{ij} = \rho_0 \frac{dx^i}{dt}\frac{dx^j}{dt}\left(\frac{dt}{ds}\right)^2$$

Energy-Momentum Tensor in Galilean Coordinates

$$\Rightarrow T^{ij} = \frac{dx^i}{dt}\frac{dx^j}{dt}\left[\frac{\rho_0}{(ds/dt)^2}\right]$$
$$\Rightarrow T^{ij} = \rho\frac{dx^i}{dt}\frac{dx^j}{dt} \tag{8}$$

This is the expression for T^{ij} referred to the Galilean coordinate system.

$$\text{Let,}\quad \frac{dx^1}{dt} = u,\quad \frac{dx^2}{dt} = v,\quad \frac{dx^3}{dt} = w,$$
$$\therefore\quad T^{ij} = \begin{bmatrix} \rho u^2 & \rho uv & \rho uw & \rho u \\ \rho vu & \rho v^2 & \rho vw & \rho v \\ \rho wu & \rho wv & \rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix}$$

Energy-Momentum Tensor for a Perfect Fluid

To derive the formula for energy momentum tensor for a perfect fluid in the form:

$$T^{\mu}_{\nu} = (\rho + p)v^{\mu}v_{\nu} - g^{\mu}_{\nu}p$$

Proof: Let $T_0^{\mu\nu}$ denote the energy momentum tensor in the proper coordinate system in which the matter is supposed to be at rest at the origin, for which:

$$T_0^{11} = T_0^{22} = T_0^{33} = p_0, \quad T_0^{44} = \rho_0 \tag{1}$$

where p_0 and ρ_0 respectively denote pressure and density of a perfect fluid in the proper coordinate system. The remaining components being all zero.
In the proper coordinate system, the Galilean coordinate system holds for which:

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 \tag{2}$$

Energy-Momentum Tensor for a Perfect Fluid

where, $c = 1$, i.e., if the motion of the fluid is considered in a gravitational system.

Let g_0^{ij} denote the fundamental tensor in the Galilean coordinate system so that:

$$g_0^{11} = g_0^{22} = g_0^{33} = -g_0^{44} = -1, \quad g_0^{ij} = 0 \text{ for } i \neq j. \tag{3}$$

Let T^{ij} and g^{ij} respectively denote the energy tensor and fundamental tensor in an arbitrary coordinate system. By the tensor law of transformation:

$$\begin{aligned} T^{ij} &= T_0^{ab} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^b} = \sum_{a=1}^{4} T_0^{aa} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} \\ &= p_0 \sum_{a=1}^{3} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} + \rho_0 \frac{\partial x^i}{\partial x_0^4} \frac{\partial x^j}{\partial x_0^4} \end{aligned} \tag{4}$$

Energy-Momentum Tensor for a Perfect Fluid

Now,

$$\begin{aligned} g^{ij} &= g_0^{ab}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^b} = \sum_{a=1}^{4} g_0^{aa}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^a} \\ &= -\sum_{a=1}^{3}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^a} + \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} \\ \Rightarrow \sum_{a=1}^{3}\frac{\partial x^i}{\partial x_0^a}\frac{\partial x^j}{\partial x_0^a} &= -g^{ij} + \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} \qquad (5) \end{aligned}$$

Using this in (4), we obtain:

$$T^{ij} = p_0\left(-g^{ij} + \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4}\right) + \rho_0\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} = (\rho_0 + p_0)\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} - p_0 g^{ij} \qquad (6)$$

Energy-Momentum Tensor for a Perfect Fluid

Since the fluid is at rest in the proper coordinate system, the velocity components can be taken as:

$$\frac{dx_0^0}{ds} = \frac{dx_0^2}{ds} = \frac{dx_0^3}{ds} = 0, \quad \frac{dx_0^4}{ds} = 1$$

$$\therefore \frac{dx^i}{ds} = \frac{\partial x^i}{\partial x_0^j}\frac{dx_0^j}{ds} = \frac{\partial x^i}{\partial x_0^4}\frac{dx_0^4}{ds} = \frac{\partial x^i}{\partial x_0^4}\cdot 1 = \frac{\partial x^i}{\partial x_0^4} \tag{7}$$

Using this in equation (6), we obtain:

$$T^{ij} = (\rho_0 + p_0)\frac{dx^i}{ds}\frac{dx^j}{ds} - p_0 g^{ij}$$

$$\Rightarrow T^{ij} = (\rho_0 + p_0)v^i v^j - p_0 g^{ij} \tag{8}$$

Energy-Momentum Tensor for a Perfect Fluid

Now,

$$
\begin{aligned}
T^i_j &= g_{jk}T^{ik} \\
\Rightarrow T^i_j &= g_{jk}\left[(\rho_0 + p_0)v^i v^k - p_0 g^{ik}\right] \\
\Rightarrow T^i_j &= (\rho_0 + p_0)v^i(g_{jk}v^k) - p_0(g_{jk}g^{ik}) \\
\Rightarrow T^i_j &= (\rho_0 + p_0)v^i v_j - p_0 g^i_j
\end{aligned}
$$

Hence,

$$
T^\mu_\nu = (p + \rho)v^\mu v_\nu - p g^\mu_\nu, \quad p_0 = p, \quad \rho_0 = \rho
$$

which is the required equation. ■

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