Geodesics - Differential Geometry

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MATHEMATICAL EXPLORATIONS

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Geodesics - Definition

Lemma:

If $g(t)$ is continuous for $0 < t < 1$ and if

$$\int_0^1 V(t)g(t)\,dt = 0$$

for all admissible functions $V(t)$ as defined above, then $g(t) = 0$.

Proof: Let if possible, there exists a value t_0 of t between 0 and 1 such that $g(t_0) \neq 0$, say $g(t_0) > 0$, then continuity of g implies that $g(t) > 0$ in some interval (a, b) such that $0 < a < t_0 < b < 1$. Further, let V be defined as

$$V(t) = \begin{cases} 0 & \text{for } 0 \leq t < a \text{ and } b < t \leq 1, \\ (t-a)^3(b-t)^3 & \text{for } a \leq t \leq b. \end{cases}$$

Geodesics - Mathematical Definition

Mathematical Definition

Mathematically, geodesics satisfy the geodesic equation, which arises from the calculus of variations when minimizing the arc length:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0 \qquad (1)$$

where:

- x^μ represents **coordinates**,
- s is the **parameter** along the curve,
- $\Gamma^\mu_{\nu\lambda}$ are **Christoffel symbols**, defining how space is curved.

Metric Tensor of a Surface

A surface in three-dimensional space can be defined as:

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \tag{2}$$

where x, y, z are functions of two independent parameters u and v. These parameters define a coordinate system on the surface.

If we make small changes in u and v, the corresponding position vector on the surface changes slightly:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv \tag{3}$$

Metric Tensor of a Surface

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv$$

This equation represents a **small displacement vector** on the surface.

- The partial derivative $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ is the **tangent vector** in the direction of **increasing** u.
- The partial derivative $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$ is the **tangent vector** in the direction of **increasing** v.

Thus, the displacement $d\mathbf{r}$ is a linear combination of these two basis vectors, weighted by the small changes du and dv.

Metric Tensor of a Surface

The squared length of the **small displacement** $d\mathbf{r}$ is given by the dot product:

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} \tag{4}$$

Substituting $d\mathbf{r}$ from above:

$$(ds)^2 = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot (\mathbf{r}_u du + \mathbf{r}_v dv) \tag{5}$$

Expanding using the **distributive property** of dot product:

$$(ds)^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(du)^2 + 2(\mathbf{r}_u \cdot \mathbf{r}_v)du dv + (\mathbf{r}_v \cdot \mathbf{r}_v)(dv)^2 \tag{6}$$

Metric Tensor of a Surface

$$(ds)^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(du)^2 + 2(\mathbf{r}_u \cdot \mathbf{r}_v)dudv + (\mathbf{r}_v \cdot \mathbf{r}_v)(dv)^2$$

The coefficients E, F, G are defined as:

$$E = \mathbf{r}_u \cdot \mathbf{r}_u \quad \text{(measures stretching in the } u\textbf{-direction)} \tag{7}$$

$$F = \mathbf{r}_u \cdot \mathbf{r}_v \quad \text{(measures how } u \textbf{ and } v \textbf{ are related}\text{, i.e., shearing)} \tag{8}$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v \quad \text{(measures stretching in the } v\textbf{-direction)}. \tag{9}$$

Thus, the equation simplifies to:

$$(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2 \tag{10}$$

This formula describes the **intrinsic geometry** of the surface and is fundamental in differential geometry.

Metric Tensor of a Surface

If a curve on the surface is given by:

$$u = u(t), \quad v = v(t) \tag{11}$$

where t **is a parameter** (e.g., time or arc length), then the differentials become:

$$du = \frac{du}{dt}dt, \quad dv = \frac{dv}{dt}dt \tag{12}$$

Substituting these into the **first fundamental form**:

$$(ds)^2 = E\left(\frac{du}{dt}dt\right)^2 + 2F\left(\frac{du}{dt}dt\right)\left(\frac{dv}{dt}dt\right) + G\left(\frac{dv}{dt}dt\right)^2 \tag{13}$$

Metric Tensor of a Surface

Dividing by $(dt)^2$,

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \tag{14}$$

This equation represents the **metric tensor of the surface**, which describes how distances are measured along the surface. It is crucial in defining geodesics because geodesics are the curves that **extremize** the length computed from this metric.

Arc Length of a Curve on a Surface

The **arc length** S along the curve is obtained by integrating the differential arc length ds over a given parameter range $t \in [0, 1]$:

$$S(\alpha) = \int_0^1 ds \qquad (15)$$

Using the given metric, ds can be rewritten as:

$$ds = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt \qquad (16)$$

Thus, the total arc length of the curve is:

$$S(\alpha) = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt \qquad (17)$$

Arc Length of a Curve on a Surface

$$S(\alpha) = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

- The expression under the square root represents the **infinitesimal squared arc length**.
- The integral sums up these small arc lengths **along the curve** from $t = 0$ to $t = 1$.
- The metric terms E, F, G depend on the surface and capture how distances **change**.
- The terms $\dot{u} = \frac{du}{dt}$ and $\dot{v} = \frac{dv}{dt}$ represent the **velocity components** along the coordinate directions.

Geodesics Differential Equation

- Let A, B be **any two points** on a given surface $r = r(u, v)$.
- Let us consider the arc which join A and B and are given by equations of the form $u = u(t)$, $v = v(t)$, where $u(t)$ and $v(t)$ are functions of class 2.
- Let us assume **without loss of generality** that for every arc α, $t = 0$ at A and $t = 1$ at B. Thus α is given by $0 \leq t \leq 1$.
- The terms $\dot{u} = \frac{du}{dt}$ and $\dot{v} = \frac{dv}{dt}$ represent the **velocity components** along the coordinate directions.

We know that,

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \tag{1}$$

Geodesics Differential Equation

Therefore, the **length** α is given by

$$S(\alpha) = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

We have the curve (arc) α **parameterized** by $u(t)$ and $v(t)$ as,

$$u = u(t), \quad v = v(t)$$

where t is a parameter (e.g., time or arc length).
Now, we slightly **deform this curve** into another curve α' , keeping the endpoints fixed at A and B. This new curve is given by:

$$u'(t) = u(t) + \epsilon g(t), \quad v'(t) = v(t) + \epsilon h(t)$$

Geodesics Differential Equation

where,
- ϵ is a **small perturbation parameter**.
- $g(t)$ and $h(t)$ are **arbitrary functions** that describe the small deformation of the curve.
- The condition $g(0) = g(1) = 0$ and $h(0) = h(1) = 0$ ensures that the endpoints remain **fixed** (i.e., the variation only affects the interior of the curve).

When we **deform** the curve, the arc length of the deformed curve α' is given by the same formula but replacing u and v with u' and v':

$$S(\alpha') = \int_0^1 \sqrt{E\dot{u'}^2 + 2F\dot{u'}\dot{v'} + G\dot{v'}^2}\, dt$$

Geodesics Differential Equation

Now we shall use the method of **calculus of variations** to find the equation of geodesic. Let,

$$f = \sqrt{2T}$$

where,

$$T(u, v, \dot{u}, \dot{v}) = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)$$

Then the difference in the **functionals** is given by:

$$S(\alpha') - S(\alpha) = \int_0^1 \left[f(u + \epsilon g, v + \epsilon h, \dot{u} + \epsilon g, \dot{v} + \epsilon h) - f(u, v, \dot{u}, \dot{v})\right] dt$$

Geodesics Differential Equation

Using **Taylor's theorem**,

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 \left[g\frac{\partial f}{\partial u} + h\frac{\partial f}{\partial v} + \dot{g}\frac{\partial f}{\partial \dot{u}} + \dot{h}\frac{\partial f}{\partial \dot{v}} \right] dt + O(\epsilon^2)$$

Using **integration by parts**,

$$\int_0^1 \dot{g}\frac{\partial f}{\partial \dot{u}} dt = \left[g\frac{\partial f}{\partial \dot{u}} \right]_0^1 - \int_0^1 g\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) dt$$

Geodesics Differential Equation

Now since $g = 0$ at $t = 0$ and $t = 1$, we get

$$\int_0^1 \dot{g} \frac{\partial f}{\partial \dot{u}} dt = -\int_0^1 g \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) dt$$

Similarly,

$$\int_0^1 \dot{h} \frac{\partial f}{\partial \dot{v}} dt = -\int_0^1 h \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right) dt$$

Geodesics Differential Equation

Thus,

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 \left[g\left(\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)\right) + h\left(\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)\right)\right] dt + O(\epsilon^2)$$

Let,

$$L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right), \quad M = \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)$$

Then,

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 (gL + hM)dt + O(\epsilon^2)$$

Geodesics Differential Equation

Thus, by definition, $S(\alpha)$ will be **stationary** and α **a geodesic if and only if** $u(t)$ are such that

$$\int_0^1 (gL + hM)dt = 0$$

for all possible g and h which satisfy $g = h = 0$ when $t = 0$ and $t = 1$.

THANK YOU

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