Material, Local and Connective Derivative :

Let a fluid particle moves from $P(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Further let $f(x, y, z, t)$ be a scalar function associated with some property of the fluid. Let the total change of f due to movement of the fluid particle from P to Q be δf. Then we have

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

Or,

$$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t} \tag{1}$$

Let,

$$\lim_{\delta t \to 0}\frac{\delta f}{\delta t} = \frac{Df}{Dt} \text{ or } \frac{df}{dt},$$

$$\lim_{\delta t \to 0}\frac{\delta x}{\delta t} = \frac{dx}{dt} = u$$

$$\lim_{\delta t \to 0}\frac{\delta y}{\delta t} = \frac{dy}{dt} = v$$

$$\lim_{\delta t \to 0}\frac{\delta z}{\delta t} = \frac{dz}{dt} = w$$

(2)

where $\vec{q} = (u, v, w)$ is the velocity of the fluid particle at P. Making $\delta t \to 0$ we get

$$\frac{Df}{Dt} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} \tag{3}$$

But,
$$\vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \tag{4}$$

and
$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \tag{5}$$

From (4) and (5) we get,

$$\vec{q}.\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{6}$$

Using (6) and (3) reduces to

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{q}.\nabla)f \tag{7}$$

Again let $g(x, y, z, t)$ be a vector function associated with some property of the fluid. Then proceeding as above, we have

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + (\vec{q}.\nabla)g \tag{8}$$

From (7) and (8), we have for both scalar and vector functions

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q}.\nabla) \tag{9}$$

$\frac{D}{Dt}$ is called the *material derivative*. The term $\frac{\partial}{\partial t}$ is called the *local derivative* and it is associated with time variation at a fixed position. The term $\vec{q}.\nabla$ is called the *connective derivative* and it is associated with the change of a physical quantity f or g due to motion of the fluid particle.

Acceleration of a Fluid Particle:

Let a fluid particle moves from $P(x, y, z)$ at time t to $Q(x + \delta x, y + \delta y, z + \delta z)$ at time $t + \delta t$.

Let, $$\vec{q} = (u, v, w) = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \tag{1}$$

be the velocity of the fluid particle at P and let $\vec{q} + \delta\vec{q}$ be the velocity of the same fluid particle at Q. Then, we have

$$\delta\vec{q} = \frac{\partial\vec{q}}{\partial x}\delta x + \frac{\partial\vec{q}}{\partial y}\delta y + \frac{\partial\vec{q}}{\partial z}\delta z + \frac{\partial\vec{q}}{\partial t}\delta t$$

Or, $$\frac{\delta\vec{q}}{\delta t} = \frac{\partial\vec{q}}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial\vec{q}}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial\vec{q}}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial\vec{q}}{\partial t} \tag{2}$$

Let,

$$\lim_{\delta t\to 0}\frac{\delta\vec{q}}{\delta t} = \frac{D\vec{q}}{Dt} \; or \; \frac{d\vec{q}}{dt},$$

$$\lim_{\delta t\to 0}\frac{\delta x}{\delta t} = \frac{dx}{dt} = u$$

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta t} = \frac{dy}{dt} = v$$

$$\lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \frac{dz}{dt} = w$$

(3)

Making $\delta t \to 0$ and using (3), (2) reduces to

$$\vec{a} = \frac{D\vec{q}}{Dt} = u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z} + \frac{\partial \vec{q}}{\partial t} \tag{4}$$

Let,
$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \tag{5}$$

From (1) and (5) we get,

$$\vec{q}.\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{6}$$

Using (6), (4) may be re-written as

$$\vec{a} = \frac{D\vec{q}}{Dt} = (\vec{q}.\nabla)\vec{q} + \frac{\partial \vec{q}}{\partial t} \tag{7}$$

which shows that the *acceleration* $\vec{a}$ of a fluid particle of fixed identity can be expressed as the material derivative of the velocity vector $\vec{q}$.

Components of acceleration in Cartesian Coordinates (x, y, z):

Let, $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$. Then

$$a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} = u\frac{\partial}{\partial x}(u\hat{\imath} + v\hat{\jmath} + w\hat{k}) + v\frac{\partial}{\partial y}(u\hat{\imath} + v\hat{\jmath} + w\hat{k}) +$$
$$w\frac{\partial}{\partial z}(u\hat{\imath} + v\hat{\jmath} + w\hat{k}) + \frac{\partial}{\partial t}(u\hat{\imath} + v\hat{\jmath} + w\hat{k})$$

$$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Ex. 1. If the velocity distribution is $\vec{q} = Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}$, where A, B, C are constants, then find the acceleration and velocity components.

Sol. The velocity distribution is $\vec{q} = Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}$, where A, B, C are constants

The acceleration $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ is given by

$$\vec{a} = \frac{\partial\vec{q}}{\partial t} + u\frac{\partial\vec{q}}{\partial x} + v\frac{\partial\vec{q}}{\partial y} + w\frac{\partial\vec{q}}{\partial z}$$

Now,

$$\vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} = Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}$$

$\therefore u = Ax^2y, v = By^2zt, w = Czt^2$

$$\frac{\partial\vec{q}}{\partial t} = \frac{\partial}{\partial t}\left(Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}\right) = By^2z\hat{\jmath} + 2Czt\hat{k}$$

$$\frac{\partial\vec{q}}{\partial x} = \frac{\partial}{\partial x}\left(Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}\right) = 2Axy\hat{\imath}$$

$$\frac{\partial\vec{q}}{\partial y} = \frac{\partial}{\partial y}\left(Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}\right) = Ax^2\hat{\imath} + 2Byzt\hat{\jmath}$$

$$\frac{\partial\vec{q}}{\partial z} = \frac{\partial}{\partial z}\left(Ax^2y\hat{\imath} + By^2zt\hat{\jmath} + Czt^2\hat{k}\right) = By^2t\hat{\jmath} + Ct^2\hat{k}$$

$$\vec{a} = By^2z\hat{\jmath} + 2Czt\hat{k} + Ax^2y.2Axy\hat{\imath} + By^2zt(Ax^2\hat{\imath} + 2Byzt\hat{\jmath}) + Czt^2(By^2t\hat{\jmath} + Ct^2\hat{k})$$

$$\vec{a} = A(2Ax^3y^2 + Bx^2y^2zt)\hat{\imath} + B(y^2z + 2By^3z^2t^2 + Cy^2zt^3)\hat{\jmath} + C(2zt + Czt^4)\hat{k}$$

The components of acceleration (a_x, a_y, a_z) are given by

$$a_x = A(2Ax^3y^2 + Bx^2y^2zt)$$

$$a_y = B(y^2z + 2By^3z^2t^2 + Cy^2zt^3)$$

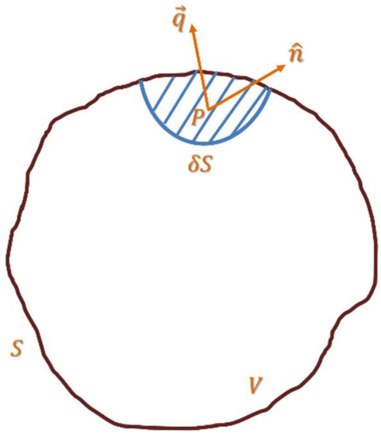
$$a_z = C(2zt + Czt^4)$$

Equation of Continuity:

The law of conservation of mass states that fluid mass can be neither created nor destroyed. Physical quantities are said to be *conserved* when they do not change with regard to time during a process. The mathematical expression of the law of conservation of mass is known as the *equation of continuity*. In continuous motion, the equation of continuity expresses the fact that the increase in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the excess of the mass that flows in over the mass that flows out.

Derivation:

Let us consider a closed surface S in a fluid medium containing a volume V fixed in space. Let $P(x,y,z)$ be any point of S and let $\rho(x,y,z,t)$ be the fluid density at P at any time t. Let, δS denote element of the surface S enclosing P. Let, $\hat{n}$ be the unit outward drawn normal at δS and let $\vec{q}$ be the fluid velocity at P.



Then the normal component of $\vec{q}$ measured outward from $V = \hat{n}.\vec{q}$

Rate of mass flow across δS per unit mass $= \rho(\hat{n}.\vec{q})\delta S$

Total rate of mas s flow across $S = \int_S \rho(\hat{n}.\vec{q})dS$

$$= \int_V \nabla.(\rho\vec{q})dV \quad \text{[By Gauss divergence theorem]}$$

Total rate of mass flow into $V = -\int_V \nabla.(\rho\vec{q})dV \qquad (1)$

Also, rate of increase of mass within $V = \frac{\partial}{\partial t}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV \qquad (2)$

Let the region V of the fluid contains neither *sources nor sinks* (i.e. there are no inlets or outlets through which fluid can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V.

Hence from (1) and (2) we get,

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla.(\rho \vec{q}) dV$$

$$=> \int_V \left[\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{q})\right] dV = 0$$

$$=> \frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{q}) = 0$$

which is known as *equation of continuity* or the conservation of mass and it holds at all points of fluid free from sources and sinks.

Cor.1. Since $\nabla.(\rho \vec{q}) = \rho \nabla.\vec{q} + \nabla \rho.\vec{q}$ therefore the equation of continuity can be expressed as

$$\frac{\partial \rho}{\partial t} + \rho \nabla.\vec{q} + \nabla \rho.\vec{q} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla.\vec{q} = 0$$

$$\frac{D}{Dt}(log\rho) + \nabla.\vec{q} = 0$$

Cor.2. For an *incompressible and heterogeneous fluid*, the density of any fluid particle is invariable with time so that $\frac{D\rho}{Dt} = 0$. Then

$$\nabla.\vec{q} = 0 \text{ i.e. div } \vec{q} = 0$$

$$\text{i.e.} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \text{if } \vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

Cor.3. For an *incompressible and homogeneous fluid*, is constant and hence $\frac{\partial \rho}{\partial t} = 0$ then

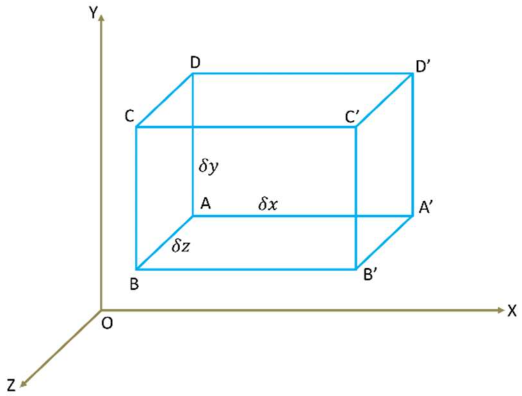
$$\nabla.(\rho \vec{q}) = 0$$

$$i.e. \nabla.\vec{q} = 0$$

$$\text{i.e.} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \text{if } \vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

Equation of Continuity in Cartesian Coordinates:

Let there be a fluid particle at $A(x, y, z)$. Let $\rho(x, y, z, t)$ be the density of the fluid at A at any time t and let u, v, w be the velocity components at A parallel to the rectangular coordinate axes. Let us construct a small parallelepiped with edges $\delta x, \delta y, \delta z$ of lengths parallel to their respective coordinate axes, having A at one of the angular points as shown in the figure.



Mass of the fluid that passes in through the face $ABCD = (\rho\delta y\delta z)u$ per unit time

$= f(x, y, z)$ (say)

Mass of the fluid that passes out through the opposite face $A'B'C'D'$

$= f(x + \delta x, y, z)$ per unit time

$= f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + \cdots$ (Expanding by Taylor's theorem)

The net gain in mass *per unit time* within the element due to flow through the faces $ABCD$ and $A'B'C'D'$ by using (1) and (2) = Mass that enters in through the face $ABCD$ – Mass that leaves through the face $A'B'C'D'$

$$= f(x, y, z) - \left[f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + \cdots\right]$$

$$= -\delta x \frac{\partial}{\partial x} f(x, y, z) \quad \text{to the first order of approximation}$$

$$= -\delta x \frac{\partial}{\partial x}(\rho u \delta y \delta z)$$

$$= -\delta x \delta y \delta z \frac{\partial}{\partial x}(\rho u)$$

Similarly, the net gain in mass *per unit time* within the element due to flow through the faces $ABA'B'$ and $CDC'D' = -\delta x \delta y \delta z \frac{\partial}{\partial y}(\rho v)$

The net gain in mass *per unit time* within the element due to flow through the faces $AA'DD'$ and $BB'CC' = -\delta x\delta y\delta z\frac{\partial}{\partial z}(\rho w)$

Total rate of mass flow into the elementary parallelepiped

$$= -\delta x\delta y\delta z\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right] \quad (1)$$

Again, the mass of the fluid within the chosen element at time t $= \rho\delta x\delta y\delta z$

Total rate of mass increase within the element $= \frac{\partial}{\partial t}(\rho\delta x\delta y\delta z)$

$$= \delta x\delta y\delta z\frac{\partial\rho}{\partial t} \quad (2)$$

Let the chosen region of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid within the element must be equal to the rate of mass flowing into the element.

Hence from (1) and (2), we have

$$\delta x\delta y\delta z\frac{\partial\rho}{\partial t} = -\delta x\delta y\delta z\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]$$

or, $$\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial v}{\partial y}+v\frac{\partial\rho}{\partial y}+\rho\frac{\partial w}{\partial z}+w\frac{\partial\rho}{\partial z}=0$$

or, $$\left[\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right]\rho+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

or, $$\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

which is the desired equation of continuity in cartesian coordinates and it holds at all point of the fluid free from sources and sinks.

Remark: If the fluid is *incompressible and homogeneous* then ρ is constant and equation of continuity reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

If the fluid is *incompressible and heterogeneous*, ρ is a function of x, y, z and t such that $\frac{D\rho}{Dt}=0$ then equation of continuity reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$