FLUID DYNAMICS

MAT 401

Source, Sink and Doublet

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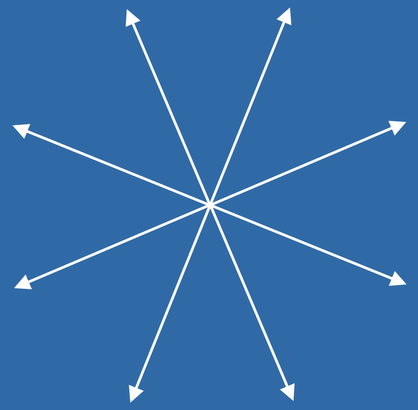
Source

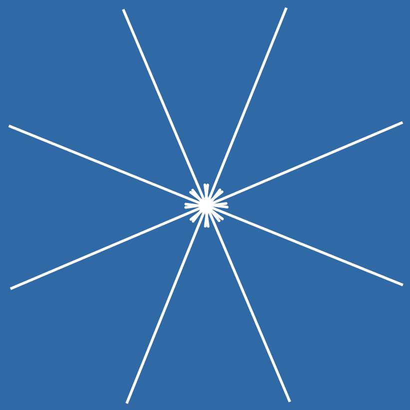
An outward symmetrical radial flow of fluid in all directions is termed as a three dimensional *source* or a *point source* or a *simple source*.



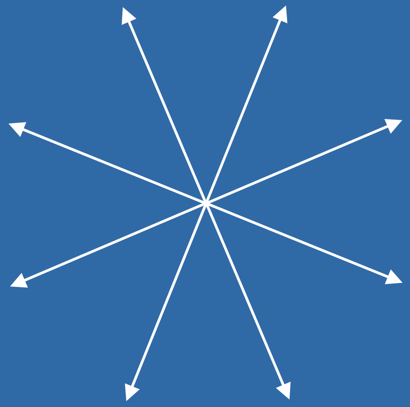
Thus, a source is a point at which fluid is *continuously created* and *distributed* e.g. an expanding bubble of gas pushing away the surrounding fluid.

Sink

A negative source is called a *sink*. At such points, the fluid is constantly moving radially *inwards* from all directions in a symmetrical manner.

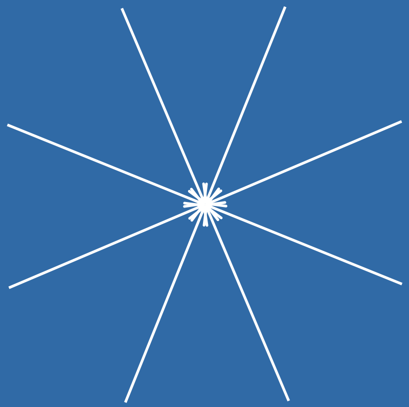


Let us consider a source at the origin. Then the *mass m* of the fluid coming out from the origin in unit time is known as the *strength of the source*.



Similarly, the amount of fluid going into the sink in a unit time is called the *strength of the sink*.

Thus a simple sink of strength m is a simple source of strength $-m$.

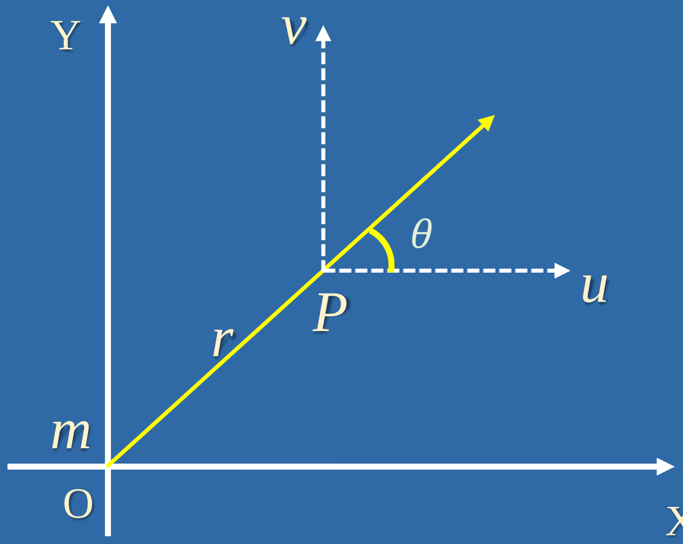


Complex Potential due to a Source:

Let us consider a source of strength m at the origin O.

To determine complex potential w at a point $P(r, \theta)$ due to this source.

The velocity of P due to the source is purely radial, let this be q_r at a distance r.

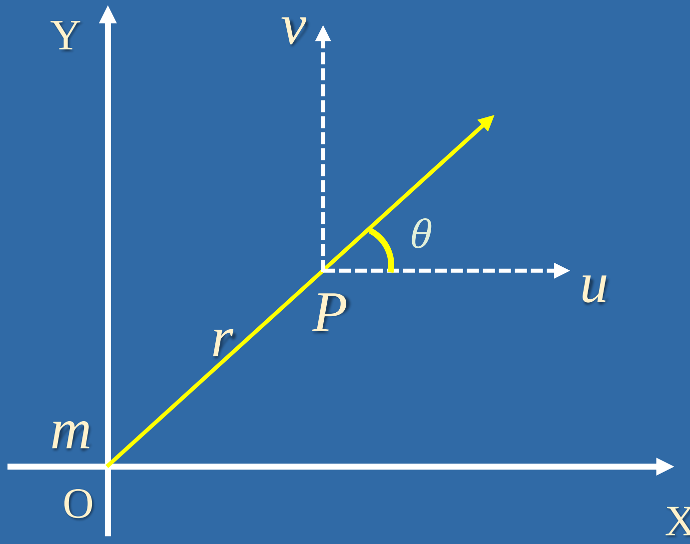


The flux across a circle of radius r surrounding the source at O is $2\pi r q_r$. Also, a source of strength m is such that the flow across any small curve surrounding it is $2\pi m$.

By conservation of mass in a steady incompressible flow, we have

$$2\pi r q_r = 2\pi m$$

$$=> q_r = \frac{m}{r}$$



Then,

$$u = q_r cos\theta = \frac{m}{r}\cos\theta$$

$$v = q_r sin\theta = \frac{m}{r}\sin\theta$$

We have, $$\frac{dw}{dz} = -u + iv = \frac{m}{r}[-cos\theta + isin\theta] = -\frac{m}{r}e^{-i\theta}$$

$$=> \frac{dw}{dz} = -\frac{m}{re^{i\theta}}$$

$$=> \frac{dw}{dz} = -\frac{m}{z}$$

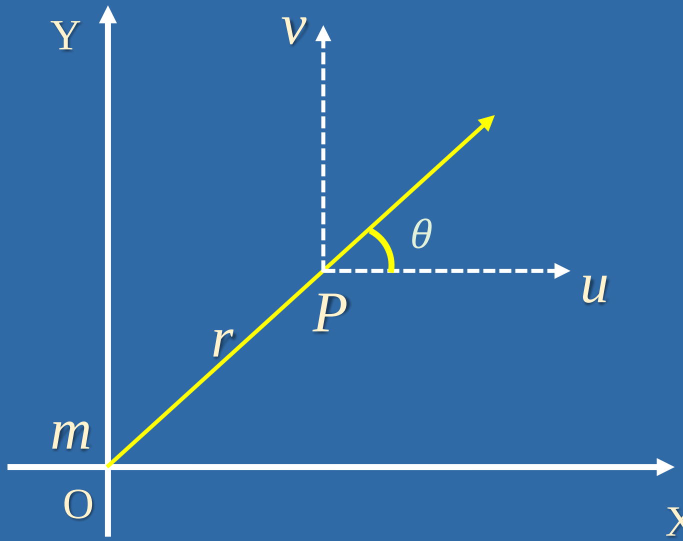
$$=> dw = -\frac{mdz}{z}$$

Integrating,

$$w = -mlogz$$

(Neglecting constant of integration)

which is the required expression for complex potential due to a source.



Deductions:

1. If the source $+m$ is at a point $z = z_1$ in place of $z = 0$; then by shifting the origin the complex potential is given by $w = -mlog(\mathrm{z} - z_1)$.

2. The complex potential w at any point z due to sources of strength $m_1, m_2, m_3, \ldots$ situated at the points $z = z_1, z_2, z_3, \ldots$ is given by

$$w = -m_1 log(\mathrm{z} - z_1) - m_2 log(\mathrm{z} - z_2) - -m_3 log(\mathrm{z} - z_3) - \cdots$$

Doublet or Dipole

A combination of a source of strength $+m$ and a sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely great and δs infinitely small but so that the product $m\delta s$ remains finite and equal to μ, is called a doublet of strength μ and the line δs taken in the sense from $-m$ to $+m$ is taken as the axis of the doublet.

Complex Potential due to a Doublet

Let, A and B be the positions of the sink and source of equal strength m and P be any point in the space.

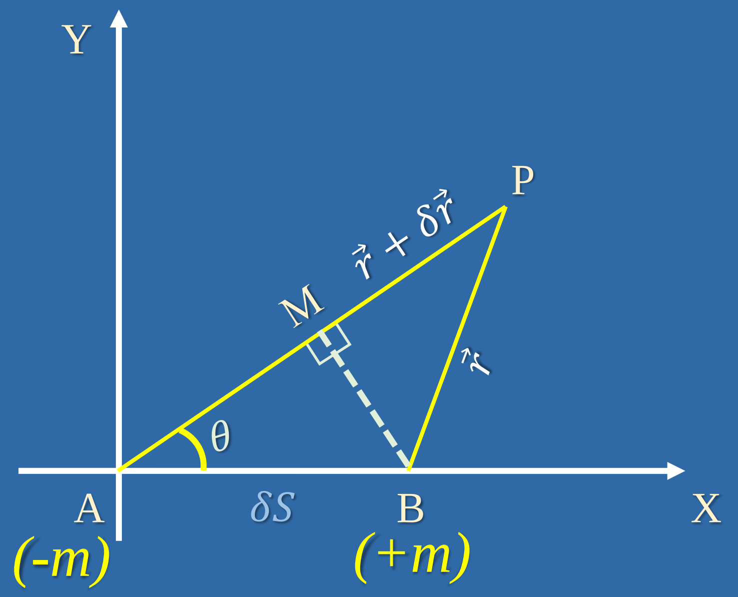
Let, $AP = \vec{r} + \delta\vec{r}$, $BP = \vec{r}$ and PAB makes angle θ with the axis of the doublet.

Let, BM be perpendicular drawn from B on AP.

Then,

$$AM = AP - MP = (\vec{r} + \delta\vec{r}) - \vec{r} = \delta\vec{r}$$

$$cos\theta = \frac{AM}{AB} = \frac{\delta r}{\delta S} => \delta r = \delta S cos\theta$$



Let, ϕ be the velocity potential due to this doublet.

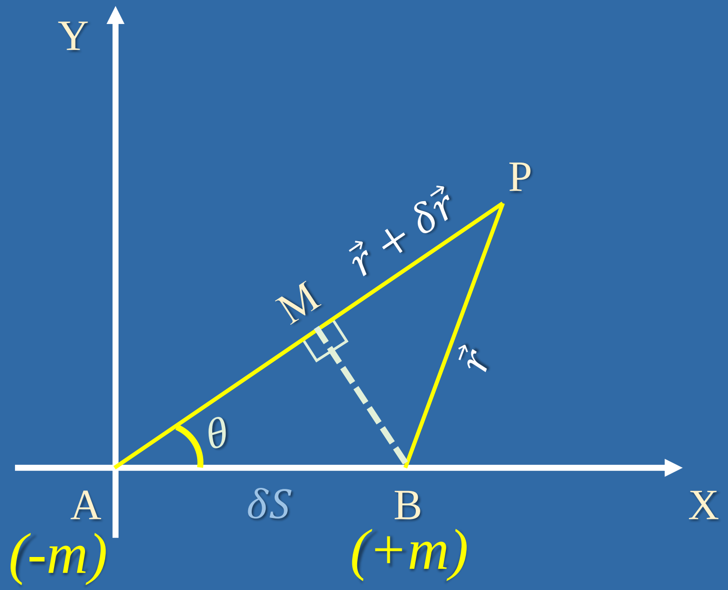
$$\phi = -mlogr - (-m)\log(r + \delta r)$$

$$= m\log\left(\frac{r + \delta r}{r}\right)$$

$$= m\log\left(1 + \frac{\delta r}{r}\right)$$

$$= \frac{m\delta r}{r} = \frac{m\delta S cos\theta}{r} = \frac{\mu cos\theta}{r} \qquad (1)$$

where $\mu = m\delta S =$ strength of the doublet



We have from Cauchy-Riemann equation,

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

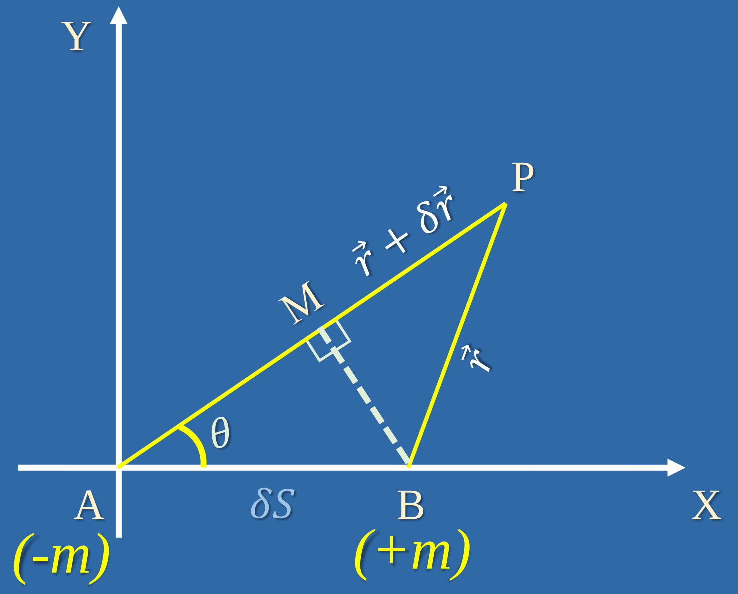
$$=> \frac{\partial}{\partial r}\left(\frac{\mu cos\theta}{r}\right) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$=> \mu cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$=> \frac{-\mu cos\theta}{r^2} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$=> \frac{\partial\psi}{\partial\theta} = \frac{-\mu cos\theta}{r}$$

$$=> \partial\psi = \frac{-\mu cos\theta}{r}\partial\theta$$



(2)

Integrating,

$$\psi = \frac{-\mu sin\theta}{r} + f(r) \qquad (3)$$

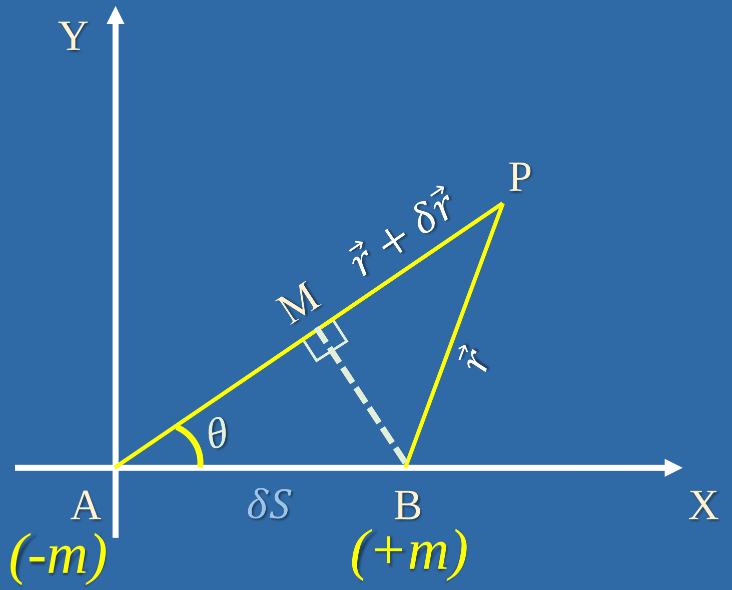
Now,

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

$$=> \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu cos\theta}{r}\right) = -\frac{\partial}{\partial r}\left[\frac{-\mu sin\theta}{r} + f(r)\right]$$

$$=> \frac{1}{r}\left(-\frac{\mu sin\theta}{r}\right) = -\left[\frac{\mu sin\theta}{r^2} + f'(r)\right]$$

$$=> -\frac{\mu sin\theta}{r^2} = -\frac{\mu sin\theta}{r^2} + f'(r) \quad => f'(r) = 0 \quad => f(r) = constant \qquad (4)$$



Omitting the additive constant equation (3) reduces to,

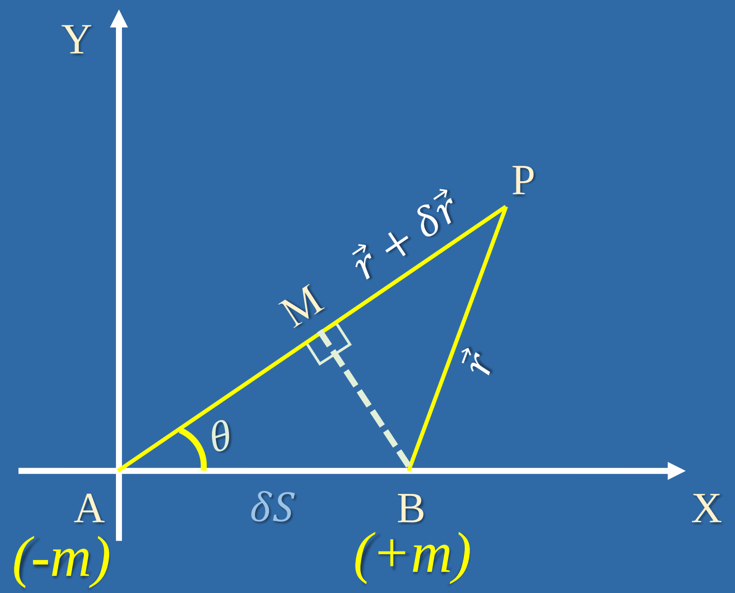
$$\psi = \frac{-\mu sin\theta}{r}$$

∴ The complex potential due to a doublet is given by

$$w = \phi + i\psi = \frac{\mu cos\theta}{r} - i\,\frac{\mu sin\theta}{r}$$

$$= \frac{\mu}{r}(cos\theta - isin\theta)$$

$$= \frac{\mu e^{-i\theta}}{r} = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z} \Rightarrow w = \frac{\mu}{z}$$



which represents the complex potential for a doublet of strength at the origin directed along x-axis.

Remark 1:

Equi-potential curves are given by

$$\phi = constant$$

$$=> \frac{\mu cos\theta}{r} = constant$$

$$=> \frac{cos\theta}{r} = C$$

$$=> rcos\theta = Cr^2$$

$$=> x = C(x^2 + y^2)$$

which represents circles touching y-axis at the origin.

Remark 2:

Streamlines are given by

$$\psi = constant$$

$$=> \frac{-\mu sin\theta}{r} = constant$$

$$=> \frac{sin\theta}{r} = C'$$

$$=> rsin\theta = C'r^2$$

$$=> y = C'(x^2 + y^2)$$

which represents circles touching x-axis at the origin.

Remark 3:

If the doublet makes an angle θ with x-axis, then θ will be replaced by $\theta - \alpha$ so that

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{i\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}$$

If the doublet be at the point $A(x', y')$ where $z' = x' + iy'$ then we have

$$w = \frac{\mu e^{i\alpha}}{z - z'}$$

Remark 4:

If the doublets of strengths $\mu_1, \mu_2, \mu_3, \dots$ situated at the points $z = z_1, z_2, z_3, \dots$ and their axes making angles $\alpha_1, \alpha_2, \alpha_3, \dots$ with x-axis, then the complex potential due to the above system is given by

$$w = \frac{\mu_1 e^{i\alpha_1}}{z - z_1} + \frac{\mu_2 e^{i\alpha_2}}{z - z_2} + \frac{\mu_3 e^{i\alpha_3}}{z - z_3} + \cdots$$