FLUID DYNAMICS

MAT 401

Bernoulli's Theorem

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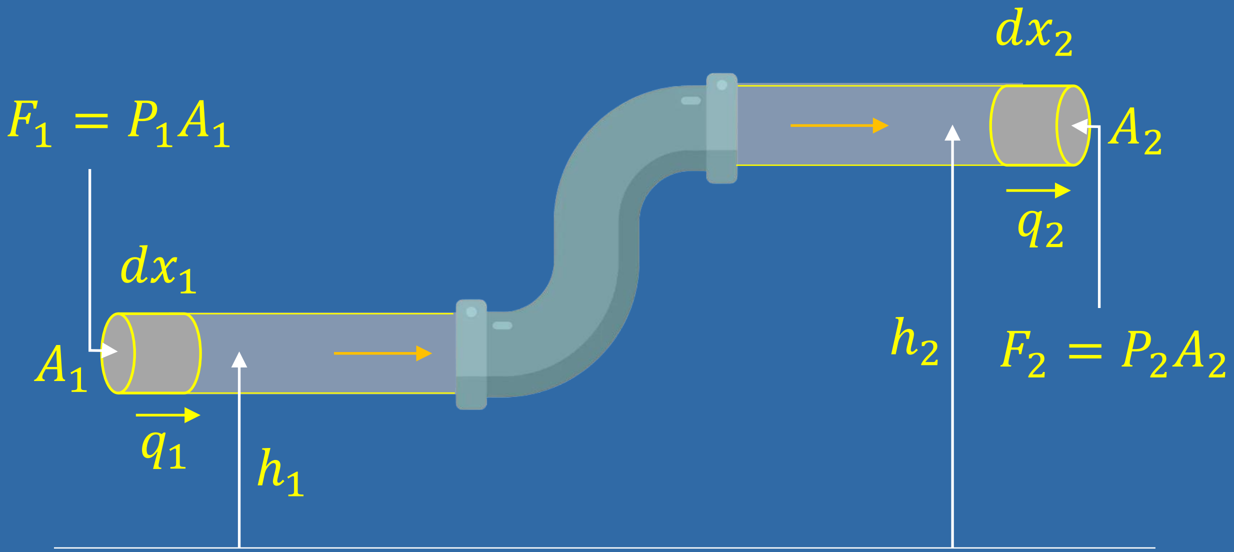
Bernoulli's theorem

Statement: Bernoulli's theorem, also known as Bernoulli's principle, states that the *whole mechanical energy* of the moving fluid, which includes gravitational potential energy of elevation, fluid pressure energy, and kinetic energy of fluid motion, *remains constant*.

Bernoulli's equation is given as follows-

$$p + \frac{1}{2}\rho q^2 + \rho g h = constant$$

where, p is the pressure exerted by the fluid
q is the velocity of the fluid
ρ is the density of the fluid
h is the height of the container



$F_1 = P_1 A_1$
dx_1
A_1
q_1
h_1
dx_2
A_2
q_2
h_2
$F_2 = P_2 A_2$

Let us consider a container in the shape of a pipe, whose two edges are placed at *different heights* and varying diameters. The relationship between the areas of cross-sections A, the flow speed q, height from the ground h, and pressure p at two different points 1 and 2 are given in the figure below.

Let us assume that the density of the incompressible fluid remains constant at both points and the energy of the fluid is conserved as there are no viscous forces in the fluid.

The work done at point 1 where the force F_1 is exerted to displace the fluid to dx_1 is

$$dW_1 = F_1 dx_1$$

Here, the force exerted at point 1 is given by, $F_1 = P_1 A_1$

where P_1 and A_1 are the pressure exerted and cross-sectional area at point 1.

$$dW_1 = P_1 A_1 \, dx_1$$

$$\Rightarrow dW_1 = P_1 \, dv$$

Similarly, the work done by the fluid at point 2 is:

$$dW_2 = P_2 A_2 \, dx_2$$

$$\Rightarrow dW_2 = P_2 \, dv$$

Now, the *total work done* by the fluid flowing through the container is,

$$dW = P_1 \, dv - P_2 \, dv$$

$$\Rightarrow dW = (P_1 - P_2) \, dv \qquad (1)$$

Now, the change in the *kinetic energy* of the fluid is given by,

$$dK = \frac{1}{2} m_2 q_2^2 - \frac{1}{2} m_1 q_1^2$$

But the mass of the fluid is

$$m = \rho dv$$

Therefore,

$$dK = \frac{1}{2}\rho dv(q_2^2 - q_1^2)$$

Similarly, the gravitational potential energy is,

$$dU = m_2 g h_2 - m_1 g h_1$$

$$dU = \rho dv g(h_2 - h_1) \qquad (1)$$

According to the law of conservation of energy,

$$dW = dK + dU$$

$$(P_1 - P_2)\, dv = \frac{1}{2}\rho dv(q_2^2 - q_1^2) + \rho dv g(h_2 - h_1)$$

$$=> (P_1 - P_2) = \frac{1}{2}\rho(q_2^2 - q_1^2) + \rho g(h_2 - h_1)$$

$$\Rightarrow P_1 + \frac{1}{2}\rho q_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho q_2^2 + \rho g h_2$$

This is Bernoulli's equation and can be expressed as

$$P + \frac{1}{2}\rho q^2 + \rho g h = \text{constant} \quad \blacksquare$$

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Motion in Two Dimension

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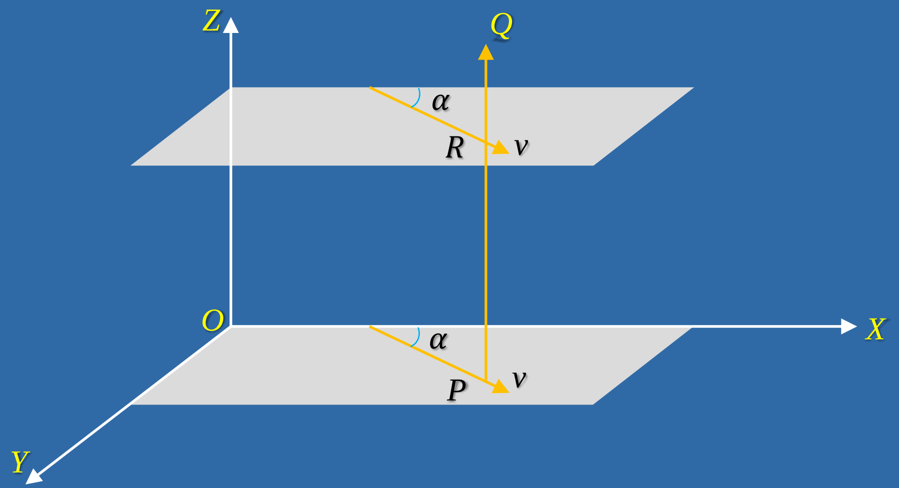
Motion in Two Dimensions

If the lines of motion are *parallel* to a fixed plane and if the velocity at corresponding points of all planes has the same magnitude and direction, then the motion is said to be *two dimensional*.

If (x, y, z) are coordinates of any point in the fluid, then all physical quantities associated with the fluid are *independent of z*. Thus u and v are functions of x, y and t and $w = 0$ for such a motion.



Stream Function or Current Function

Let u and v be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow or streamline is

$$\frac{dx}{u} = \frac{dy}{v}$$

$$=> vdx - udy = 0 \qquad (1)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 => \frac{\partial v}{\partial y} = \frac{\partial(-u)}{\partial x} \qquad (2)$$

Equation (2) shows that LHS of equation (1) must be an exact differential, $d\psi$ (say). Thus we have,

$$vdx - udy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \qquad (3)$$

so that,

$$u = -\frac{\partial \psi}{\partial y} \qquad\qquad v = \frac{\partial \psi}{\partial x} \qquad (4)$$

This function ψ is known as the *stream function*.

Using equations (1) and (3), the streamlines are given by

$$d\psi = 0 => \psi = c, \quad \text{where c is arbitrary constant.}$$

Thus the stream function is *constant* along a streamline.

Physical Interpretation of Stream Function

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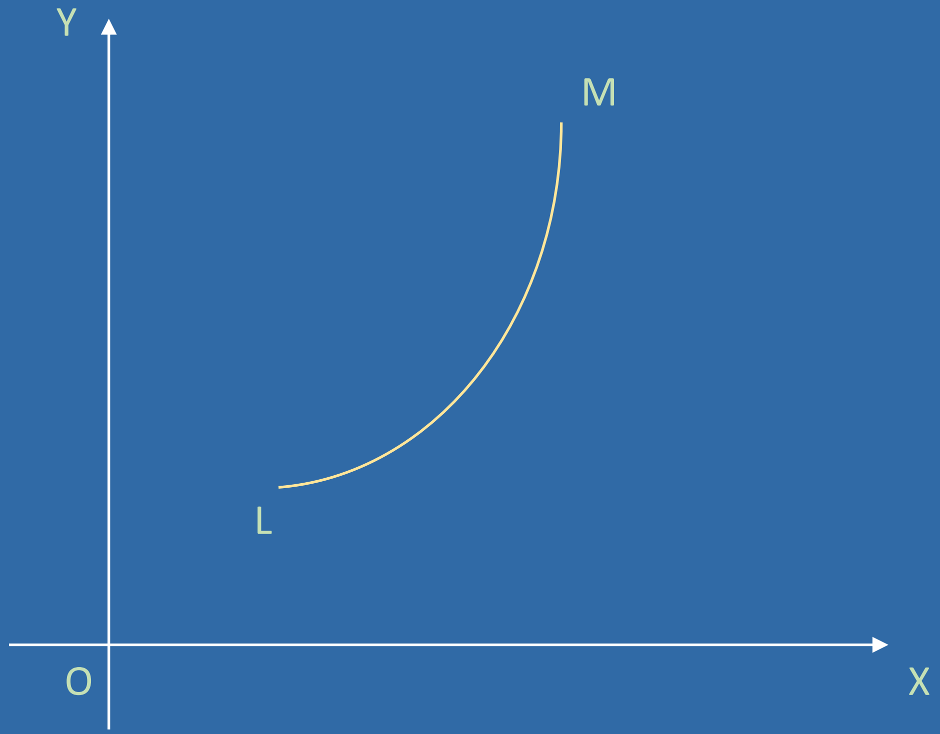
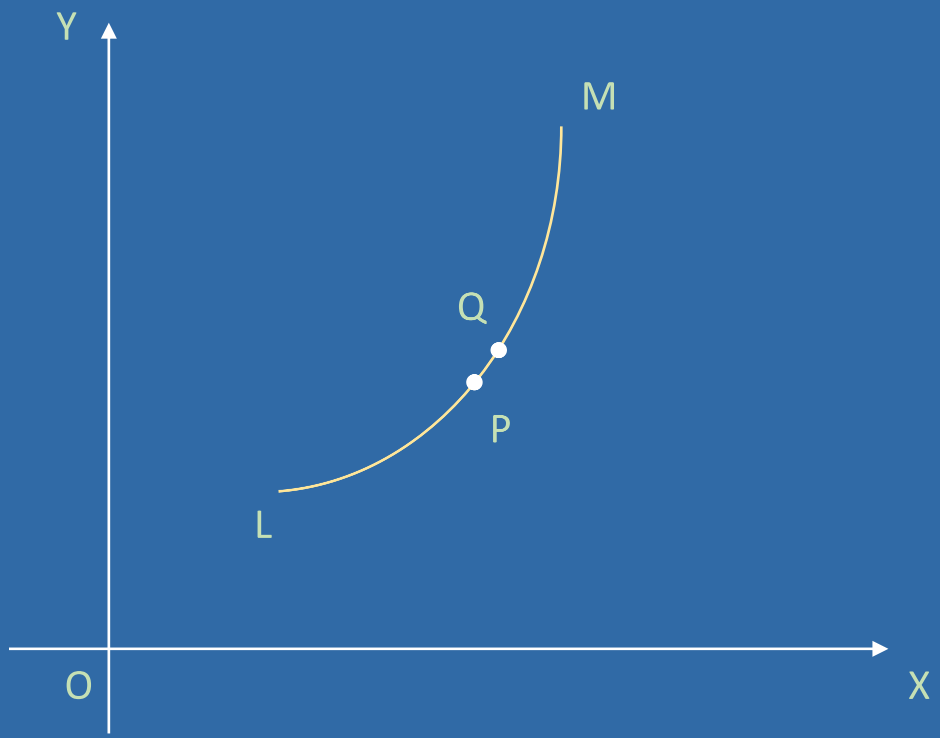
Let LM be any curve in *XY-plane* and let ψ_1 and ψ_2 be the stream functions at L and M respectively.



Let LM be any curve in *XY-plane* and let ψ_1 and ψ_2 be the stream functions at L and M respectively.

Let P be an arbitrary point on LM such that *arc* $LP = s$ and let Q be a neighbouring point on LM such that *arc* $LQ = s+\delta s$.



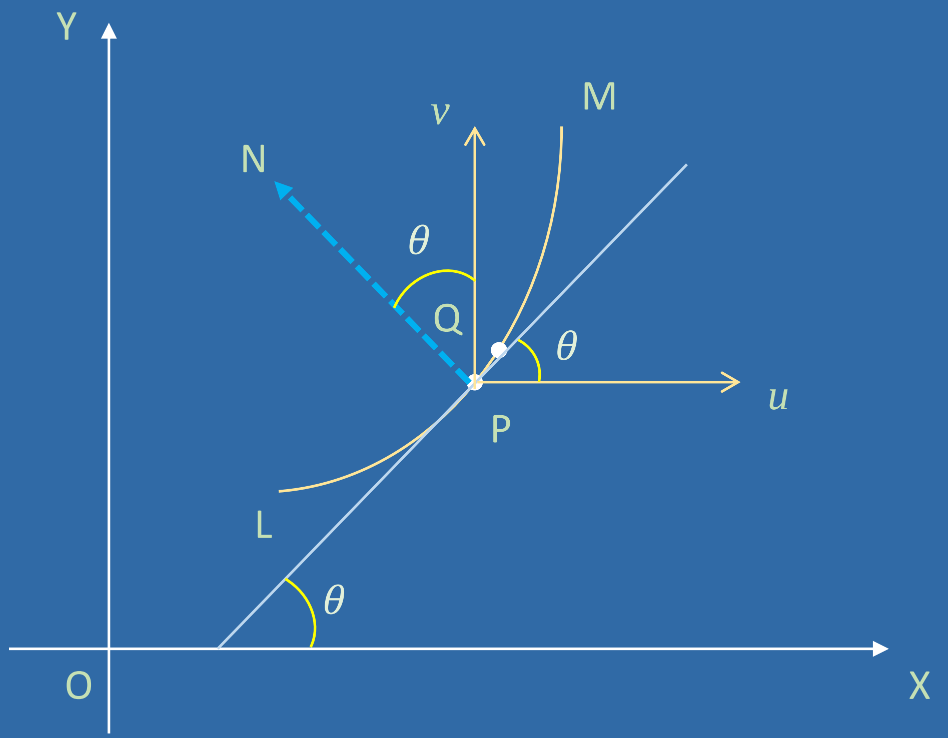
Let LM be any curve in *XY-plane* and let ψ_1 and ψ_2 be the stream functions at L and M respectively.

Let P be an arbitrary point on LM such that *arc LP = s* and let Q be a neighbouring point on LM such that *arc LQ = s+δs*.

Let θ be be the angle between tangent at P and x-axis.

If u and v be the velocity components at P then the direction cosines of the normal at P are

$$(\cos(90+\theta), cos\theta, 0) \text{ i.e. } (-sin\theta, cos\theta, 0)$$



The flow across the curve from right to left is $= \int_{LM} \vec{q}.\hat{n}ds$

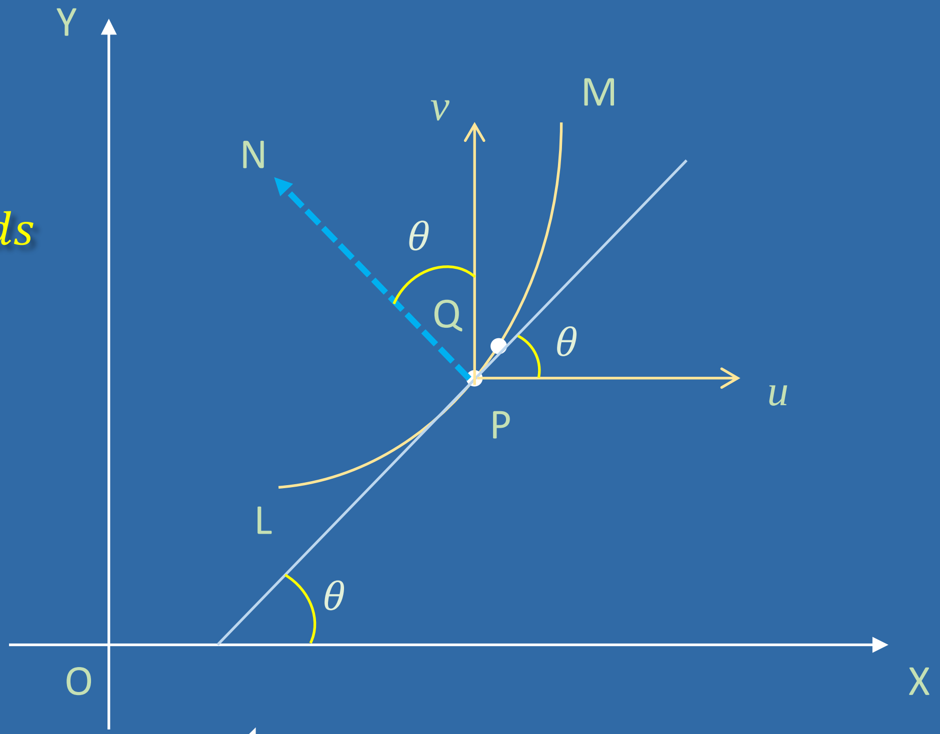
$$= \int_{LM} (u\hat{\imath} + v\hat{\jmath}).(-sin\theta\hat{\imath} + cos\theta\hat{\jmath})ds$$

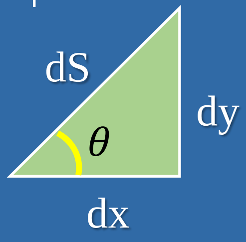
$$= \int_{LM} (-usin\theta + vcos\theta)ds \qquad (1)$$

When ψ is stream function then we have

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}$$

$$cos\theta = \frac{dx}{ds}, \qquad sin\theta = \frac{dy}{ds}$$





$$\hat{n} = -sin\theta\hat{\imath} + cos\theta\hat{\jmath}$$
$$\vec{q} = u\hat{\imath} + v\hat{\jmath}$$

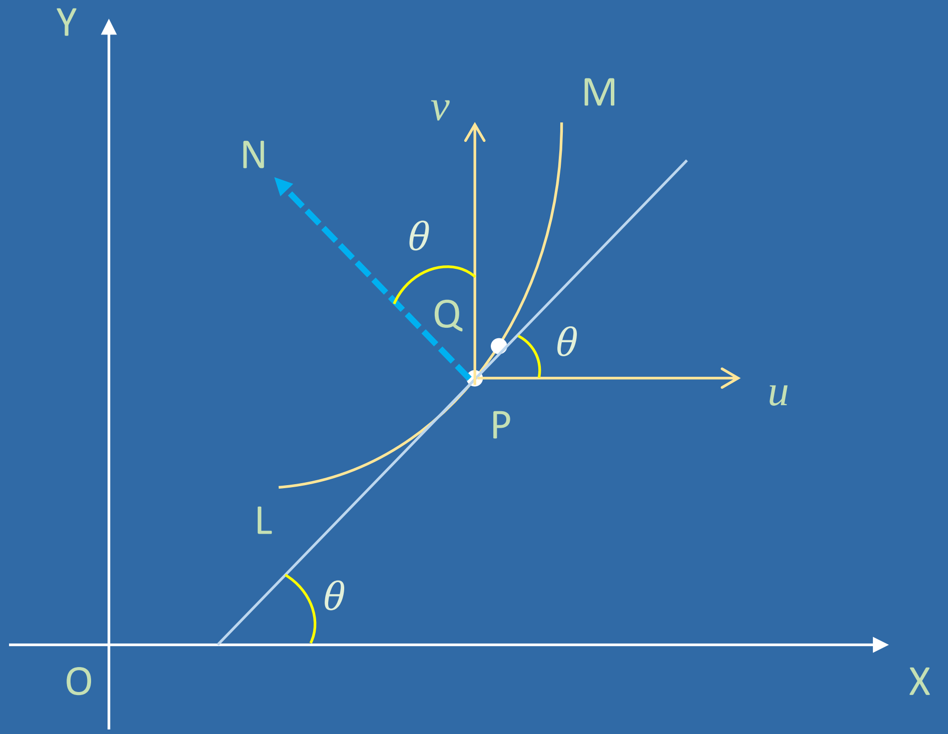
$$=> (1) = \int_{LM} \left(\frac{\partial \psi}{\partial y} sin\theta + \frac{\partial \psi}{\partial x} cos\theta \right) ds$$

$$= \int_{LM} \left(\frac{\partial \psi}{\partial y} \frac{dy}{ds} + \frac{\partial \psi}{\partial x} \frac{dx}{ds} \right) ds$$

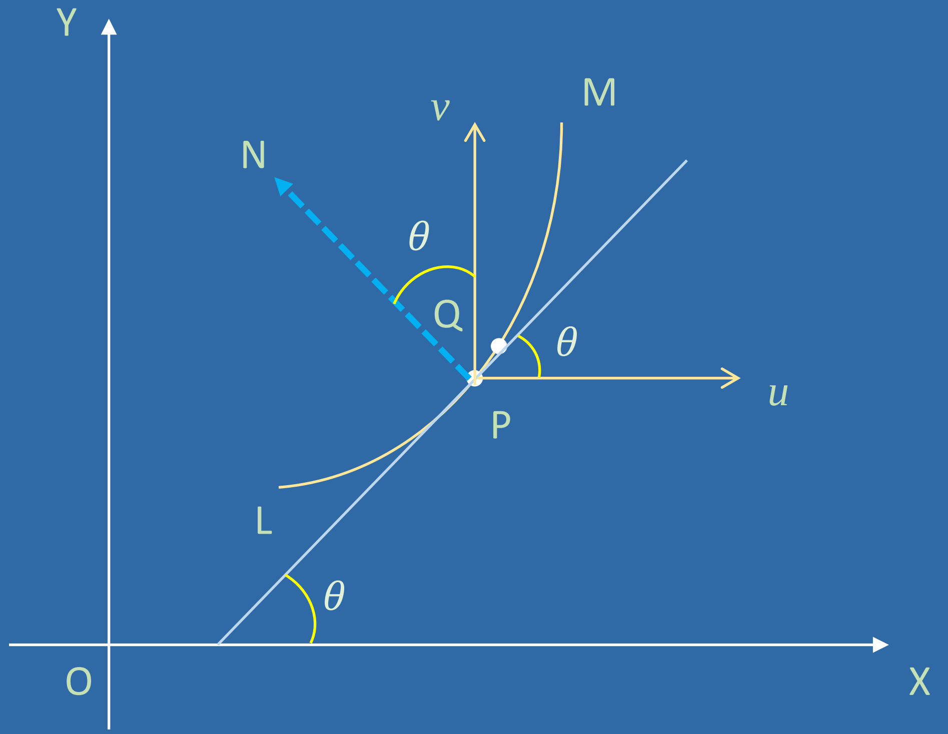
$$= \int_{LM} \left(\frac{\partial \psi}{\partial x} \frac{dx}{ds} + \frac{\partial \psi}{\partial y} \frac{dy}{ds} \right) ds$$

$$= \int_{LM} \left(\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \right)$$

$$= \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$



where ψ_1 and ψ_2 are the values of ψ at the *initial* and *final* points of the curve. Thus the difference of the values of a stream function at any two points represents the flow across that curve, joining the two points.



Complex Potential

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Complex Potential

Let, $w = \phi + i\psi$ be taken as a function of $x + iy$ i.e., z.

Let, $w = f(z)$ i.e. $\phi + i\psi = f(x + iy)$ (1)

Differentiating equation (1) w.r.t. x and y respectively, we get

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = f'(x + iy) \quad (2)$$

and

$$\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = if'(x + iy)$$

$$=> \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = i\left\{\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right\}$$

Equating real and imaginary parts we get,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \quad (3)$$

which are *Cauchy-Riemann equations*. Then *w* is an analytic function of *z* and *w* is known as the *complex potential*.

Conversely, if w is an analytic function of z, then its real part is the *velocity potential* and imaginary part is the *stream function* of an irrotational two-dimensional motion.