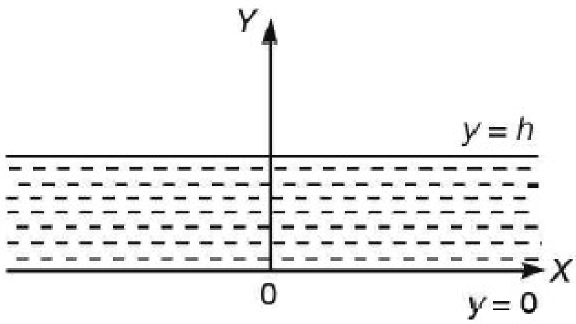
Steady laminar flow between two parallel plates. Plane Couette flow

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word 'infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two-dimensional (i.e. $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel. Here we take the velocity components v and w to be zero everywhere. Moreover, the flow being steady, the flow variables are independent of time ($\partial/\partial t = 0$). Furthermore, the equation of continuity



$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right) \tag{1}$$

reduces to $\partial u/\partial x = 0$ so that $u = u(y)$. Thus for the present problem, we have

$$u = u(y), \quad v = 0, \quad w = 0, \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial t} = 0. \tag{2}$$

For the present two-dimensional flow in the absence of body forces, the Navier-Stokes equations for x and y-directions are:

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{d^2u}{dy^2}\right) \tag{3}$$

$$0 = -\frac{\partial p}{\partial y} \tag{4}$$

Equation (4) shows that the pressure does not depend on y. Hence, p is a function of x alone, and so equation (3) reduces to

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\left(\frac{dp}{dx}\right) \tag{5}$$

Differentiating both sides of (5) with respect to x:

$$0 = \frac{1}{\mu}\frac{d^2p}{dx^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0. \tag{6}$$

so that

$$\frac{dp}{dx} = \text{const.} = P \quad \text{(say)}. \tag{7}$$

Then, (5) reduces to

$$\frac{d^2u}{dy^2} = \frac{P}{\mu} \tag{8}$$

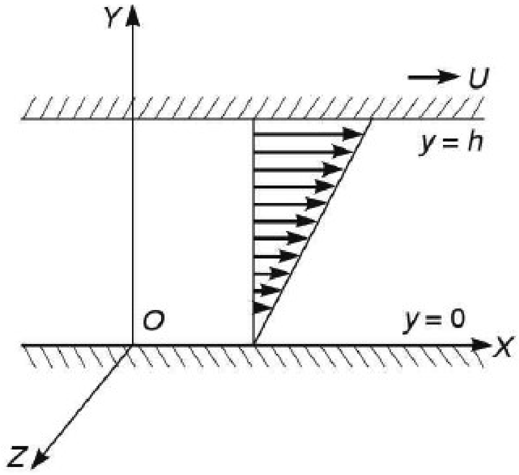
Integrating (8),

$$\frac{du}{dy} = \frac{Py}{\mu} + A \tag{9}$$

Integrating (9),

$$u = Ay + B + \frac{P}{2\mu}y^2 \tag{10}$$

where A and B are arbitrary constants to be determined by the boundary conditions of the problem under consideration.



For the plane Couette flow, $P = 0$. Again, the plate $y = 0$ is kept at rest and the plate $y = h$ is allowed to move with velocity U. Then the no-slip condition gives rise to the boundary conditions:

$$u = 0 \quad \text{at} \quad y = 0; \quad \text{and} \quad u = U \quad \text{at} \quad y = h. \tag{11}$$

Using (11), (10) yields

$$0 = B \quad \text{and} \quad U = Ah + B \tag{12}$$

so that

$$B = 0 \quad \text{and} \quad A = \frac{U}{h} \tag{13}$$

Using (13), (10) yields

$$u = \frac{Uy}{h} \tag{14}$$

The velocity distribution is linear as shown in the adjoining figure. Now, the skin friction (or drag per unit area), i.e., the shearing stress at the plates σ_{yx} is given by:

$$\sigma_{yx} = \mu\left(\frac{du}{dy}\right) = \mu\frac{U}{h}$$