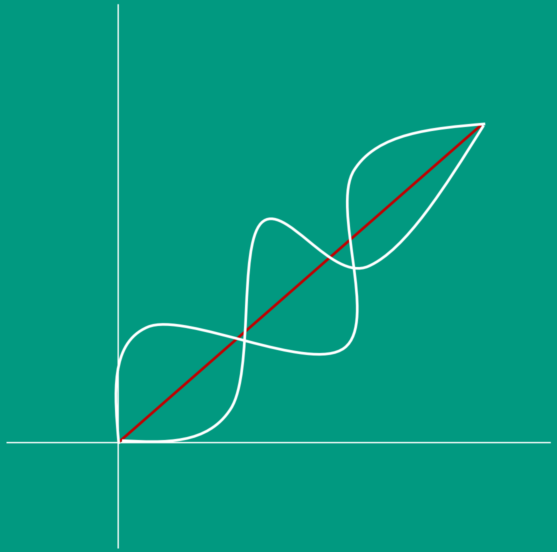
GEOMETRICAL AND DIFFERENTIAL CONSTRAINT
POSSIBLE AND VIRTUAL DISPLACEMENT
POSSIBLE ACCELERATION

MAT 103



FINITE OR GEOMETRICAL CONSTRAINTS

The constraints which can be expressed as $f(\vec{r_1}, \vec{r_2}, \ldots \vec{r_N}, t) = 0$ are called *finite or geometrical constraints*. These constraints imposed restrictions on the *position* of the particle but do not impose restriction on velocity. At any instant of time, the particle cannot posses any arbitrary position.

If t represents time, $\vec{r_k}(k = 1,2, \ldots N)$ represents the position vector, $\dot{\vec{r_k}} = \frac{d\vec{r_k}}{dt} = \vec{v_k}(k = 1,2, \ldots N)$ represents the velocity of the particle P_k of the system then *finite or geometrical constraint* is

$$f(\vec{r_k}, t) = 0 \qquad \text{where } f(\vec{r_k}, t) = f(\vec{r_1}, \vec{r_2}, \ldots \vec{r_N}, t)$$

DIFFERENTIAL OR KINEMATICAL CONSTRAINTS

The constraints which can be expressed as $f\left(\overrightarrow{r_1}, \overrightarrow{r_2}, \dots \overrightarrow{r_N}, \dot{\overrightarrow{r_1}}, \dot{\overrightarrow{r_2}}, \dots, \dot{\overrightarrow{r_N}}, t\right) = 0$ are called *differential or kinematical constraints*. These constraints imposed restrictions on the *velocity* of the particle but do not impose restriction on the position of the particle. At any instant of time, the particle can posses any arbitrary position. If t represents time, $\overrightarrow{r_k}(k = 1,2, \dots N)$ represents the position vector The differential or kinematical constraints are

$$f\left(\overrightarrow{r_k}, \dot{\overrightarrow{r_k}}, t\right) = 0$$

where $f\left(\overrightarrow{r_k}, \dot{\overrightarrow{r_k}}, t\right) = f\left(\overrightarrow{r_1}, \overrightarrow{r_2}, \dots \overrightarrow{r_N}, \dot{\overrightarrow{r_1}}, \dot{\overrightarrow{r_2}}, \dots, \dot{\overrightarrow{r_N}}, t\right)$

If t represents time, $\overrightarrow{r_k}(k=1,2,\ldots N)$ represents the position vector, $\dot{\overrightarrow{r_k}} = \frac{d\overrightarrow{r_k}}{dt} = \overrightarrow{v_k}(k=1,2,\ldots N)$ represents the velocity of the particle P_k of the system then differential or kinematical constraint is

$$f\left(\overrightarrow{r_k}, \dot{\overrightarrow{r_k}}, t\right) = 0 \quad (1)$$

where $f\left(\overrightarrow{r_k}, \dot{\overrightarrow{r_k}}, t\right) = f\left(\overrightarrow{r_1}, \overrightarrow{r_2}, \ldots \overrightarrow{r_N}, \dot{\overrightarrow{r_1}}, \dot{\overrightarrow{r_2}}, \ldots, \dot{\overrightarrow{r_N}}, t\right)$

and the finite and geometrical constraint is

$$f(\overrightarrow{r_k}, t) = 0 \quad (2)$$

where $f(\overrightarrow{r_k}, t) = f(\overrightarrow{r_1}, \overrightarrow{r_2}, \ldots \overrightarrow{r_N}, t)$

Given a finite constraint of type (2). A system cannot occupy an *arbitrary position* in space at every given instant of time because finite constraints impose restrictions to *possible positions* of the system at time t.

In differential constraint the system may occupy any *arbitrary position* in space at any time t.

In this position the velocity of the particles of the system cannot any longer be *arbitrary* because the differential constraint impose restrictions on these velocities.

Now, we will consider only such differential constraints whose equation contain the velocity of the particles in *linear form* as

$$\sum_{k=1}^{N} \overrightarrow{l_k}.\dot{\overrightarrow{r_k}} + D = 0 \tag{3}$$

where the vectors $\overrightarrow{l_k}(k = 1,2,\dots,N)$ and the scalar D are specified functions t and $\overrightarrow{r_1}, \overrightarrow{r_2}, \dots \overrightarrow{r_N}$.

Each finite constraint of the form

$$f(\overrightarrow{r_k}, t) = 0 \tag{4}$$

Implies a differential constraint whose equation is obtained by termwise differentiation of equation (4) as

$$\sum_{k=1}^{N} \frac{\partial f}{\partial \overrightarrow{r_k}} . \dot{\overrightarrow{r_k}} + \frac{\partial f}{\partial t} = 0 \qquad (5)$$

where $\overrightarrow{r_k} = x_k \hat{\imath} + y_k \hat{\jmath} + z_k \hat{k}$

$$\frac{\partial f}{\partial \overrightarrow{r_k}} = \frac{\partial f}{\partial \overrightarrow{x_k}} \hat{\imath} + \frac{\partial f}{\partial \overrightarrow{y_k}} \hat{\jmath} + \frac{\partial f}{\partial \overrightarrow{z_k}} \hat{k}$$

But a differential constraint of type (5) is not equivalent to the finite constraint (4) because it is equivalent to the finite constraint

$$f(\overrightarrow{r_k}, t) = c$$

where c is an arbitrary constant. So the finite constraint (5) is integrable.

In rectangular cartesian coordinates the constraint equations are

$$f\left(\overrightarrow{r_k}, \dot{\overrightarrow{r_k}}, t\right) = 0 \qquad f(\overrightarrow{r_k}, t) = 0$$

$$\sum_{k=1}^{N} \overrightarrow{l_k}.\dot{\overrightarrow{r_k}} + D = 0 \qquad \sum_{k=1}^{N} \frac{\partial f}{\partial \overrightarrow{r_k}}.\dot{\overrightarrow{r_k}} + \frac{\partial f}{\partial t} = 0$$

are written as

$$f(x_k, y_k, z_k, \dot{x_k}, \dot{y_k}, \dot{z_k}, t) = 0$$

$$f(x_k, y_k, z_k, t) = 0$$

$$\sum_{k=1}^{N} (A_k, \dot{x_k} + B_k, \dot{y_k} + C_k, \dot{z_k}) + D = 0$$

$$\sum_{k=1}^{N} \left(\frac{\partial f}{\partial x_k}.\dot{x_k} + \frac{\partial f}{\partial y_k}\dot{y_k} + \frac{\partial f}{\partial z_k}\dot{z_k} \right) + \frac{\partial f}{\partial t} = 0$$

where $\overrightarrow{r_k} = x_k\hat{\imath} + y_k\hat{\jmath} + z_k\hat{k} \qquad \overrightarrow{l_k} = A_k\hat{\imath} + B_k\hat{\jmath} + C_k\hat{k}$

and A_k, B_k, C_k $(k = 1,2, \dots, N)$ are scalar functions of $x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_N, y_N, z_N$.

The finite constraint $f(\overrightarrow{r_k}, t) = 0$ is called *stationary* if time (t) is not expressed explicitly in the constraint equation i.e. $\frac{\partial f}{\partial t} = 0$.

The differential constraint $\sum_{k=1}^{N} \frac{\partial f}{\partial \overrightarrow{r_k}} . \dot{\overrightarrow{r_k}} + \frac{\partial f}{\partial t} = 0$ is called *stationary* if $\frac{\partial f}{\partial t} = 0$ i.e. $\sum_{k=1}^{N} \frac{\partial f}{\partial \overrightarrow{r_k}} . \dot{\overrightarrow{r_k}} = 0$

The differential constraint

$$\sum_{k=1}^{N} \overrightarrow{l_k} . \dot{\overrightarrow{r_k}} + D = 0$$

is stationary if $D = 0$ and the vectors $\overrightarrow{l_k}$ and coefficients A_k, B_k, C_k are not explicit function of t.

POSSIBLE AND VIRTUAL DISPLACEMENT

Let us consider a system with *d finite constraints*

$$f_\alpha(\vec{r}_k, t) = 0 \qquad (\alpha = 1,2,3, \dots, d) \qquad (1)$$

and *g differential constraints*

$$\sum_{k=1}^{N} \vec{l}_{\beta k} . \dot{\vec{r}_k} + D_\beta = 0 \qquad (\beta = 1,2,3, \dots, g)$$

$$=> \sum_{k=1}^{N} \vec{l}_{\beta k} . \overrightarrow{v_k} + D_\beta = 0 \qquad (2)$$

We replace finite constraints (1) by differential constraints, obtained by differentiating w.r.t. time ‘t’ as

$$\sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}} . \dot{\overrightarrow{r_k}} + \frac{\partial f_\alpha}{\partial t} = 0 \;=> \sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}} . \overrightarrow{v_k} + \frac{\partial f_\alpha}{\partial t} = 0 \quad (\alpha = 1,2,3, \dots, d) \qquad (3)$$

The system of vectors $\overrightarrow{v_k}$ are called *possible velocities* for a certain instant of time t and for a certain possible position of the system if the vectors $\overrightarrow{v_k}$ satisfy *(d+g)* linear equations (2) and (3).

For every possible position of the system at time *t* ∃ an infinity of systems of possible velocities.

Let us consider the system of infinite possible displacements

$$d\vec{r}_k = \vec{v}_k dt \qquad (k = 1,2,3, \dots, N) \qquad (4)$$

where $\vec{v}_k' s$ are the possible velocities.

These displacements are called *possible displacements*.

Multiplying equation (2) and (3) termwise by dt we get,

$$\sum_{k=1}^{N} \vec{l}_{\beta k} . \overrightarrow{v_k}\, dt + D_\beta dt = 0 \qquad (5)$$

$$\sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}} . \overrightarrow{v_k} dt + \frac{\partial f_\alpha}{\partial t} dt = 0 \qquad (6)$$

Using equations (4) we have

$$\sum_{k=1}^{N} \vec{l}_{\beta k} . d\vec{r}_k + D_\beta dt = 0 \qquad (7)$$

$$\sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}} . d\vec{r}_k + \frac{\partial f_\alpha}{\partial t} dt = 0 \qquad (8)$$

These equations determine the possible displacements.

Let us take two systems of possible displacements at the same instant of time and same position of the system. Then

$$d\vec{r}_k = \vec{v}_k dt \qquad\qquad d\vec{r}_k{}' = \vec{v}_k{}' dt \tag{9}$$

Both $d\vec{r}_k$ and $d\vec{r}_k{}'$ satisfy the equations (7) and (8). So the difference

$$\delta\vec{r}_k = d\vec{r}_k' - d\vec{r}_k \qquad\qquad (k = 1,2,3,\dots,N)$$

satisfy the homogeneous relations

$$\sum_{k=1}^{N} \vec{l}_{\beta k}.\delta\vec{r}_k = 0 \qquad\qquad \sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}}.\delta\vec{r}_k = 0$$

The difference $\delta\vec{r}_k = d\vec{r}_k' - d\vec{r}_k$ are called *virtual displacements*.

POSSIBLE ACCELERATION

Let us consider a system of particles $P_k(k = 1,2, \ldots, N)$ with virtual displacement $\delta\vec{r}_k(k = 1,2, \ldots, N)$.

Let $\vec{F}_k$ be the corresponding forces impressed at the point $P_k(k = 1,2, \ldots, N)$ of the system.

Case I: If the *constraints were absent*.

By the Newton's second law of motion we have,

$$\vec{F}_k = m_k\vec{\omega}_k \qquad (k = 1,2, \ldots, N)$$

where m_k is the mass of the particle P_k and $\vec{\omega}_k$ is the acceleration of the particle P_k.

Case II: If the *constraints are present*.

The acceleration is given by

$$\vec{F}_k = m_k\vec{\omega}_k \Rightarrow \vec{\omega}_k = \frac{\vec{F}_k}{m_k} \qquad (k = 1,2, \ldots, N) \qquad (1)$$

At a given instant of time *'t'* in a given position $\vec{r}_k$ of the particle of the system and for a given velocities $\overrightarrow{v_k}$.

The differential constraints $(\alpha = 1,2,3, \dots, d)$ are given by

$$\sum_{k=1}^{N} \vec{l}_{\beta k}.\overrightarrow{v_k} + D_\beta = 0 \qquad (2)$$

$$\sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}}.\overrightarrow{v_k} + \frac{\partial f_\alpha}{\partial t} = 0 \qquad (3)$$

Differentiating equations (2) and (3) w.r.t. time *'t'* we get,

$$\sum_{k=1}^{N} \vec{l}_{\beta k}.\overrightarrow{\omega_k} + \sum_{k=1}^{N} \frac{d}{dt}\left(\vec{l}_{\beta k}\right)\vec{l}_k + \frac{d}{dt}\left(D_\beta\right) = 0$$

and

$$\sum_{k=1}^{N} \frac{\partial f_\alpha}{\partial \overrightarrow{r_k}}.\overrightarrow{\omega_k} + \sum_{k=1}^{N} \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \overrightarrow{r_k}}\right).\overrightarrow{v_k} + \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial t}\right) = 0$$

These are expressions for the restrictions imposed by the constraints on the acceleration $\overrightarrow{\omega_k}$ of the particle of the system.

The accelerations $\vec{\omega}_k = \frac{\vec{F}_k}{m_k}$ may not satisfy these relations. Then the constraints will act on the particles P_k of the system with certain supplementary forces $\vec{R}_k (k = 1,2, \dots, N)$. $\vec{R}_k$ is known as the *reaction forces* of the constraints. Then the accelerations are determined from the equations

$$m_k \vec{\omega}_k = \vec{F}_k + \vec{R}_k$$

where $\vec{F}_k$ is called the effective forces and $\vec{F}_k = \vec{F}_k(\vec{r}_k, \vec{v}_k, t)$.