SPECIAL THEORY OF RELATIVITY

MAT 305

LORENTZ TRANSFORMATION FORMS A GROUP

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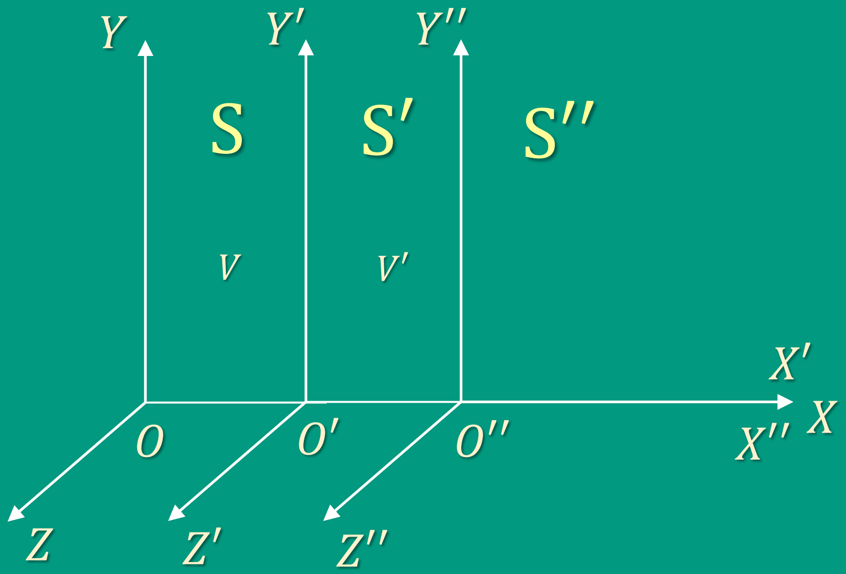
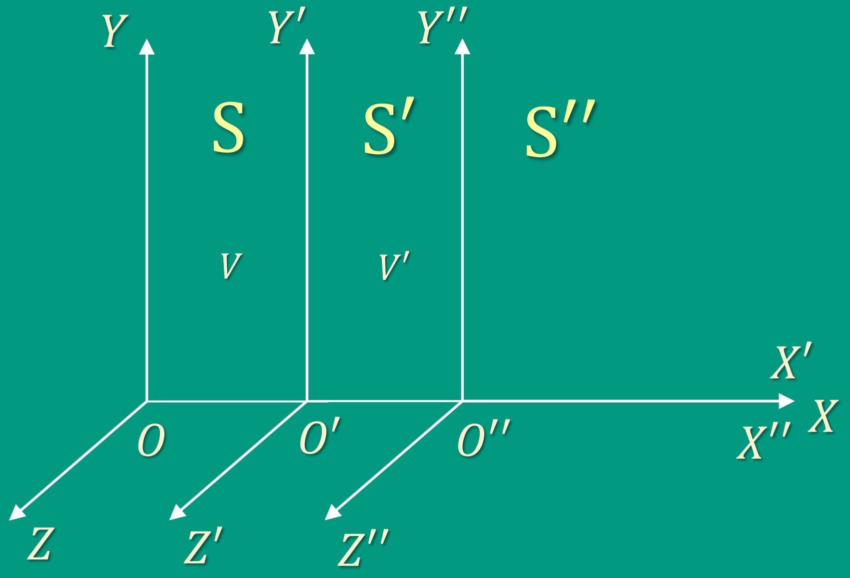
Let us consider three inertial frame of references S and S' and S'' having cartesian coordinate axes as XYZ, $X'Y'Z'$ and $X''Y''Z''$ and origins O, O' and O'' respectively.

At time $t = t' = t'' = 0$, all the frames are at rest so that their origins O, O' and O'' coincide with each other.



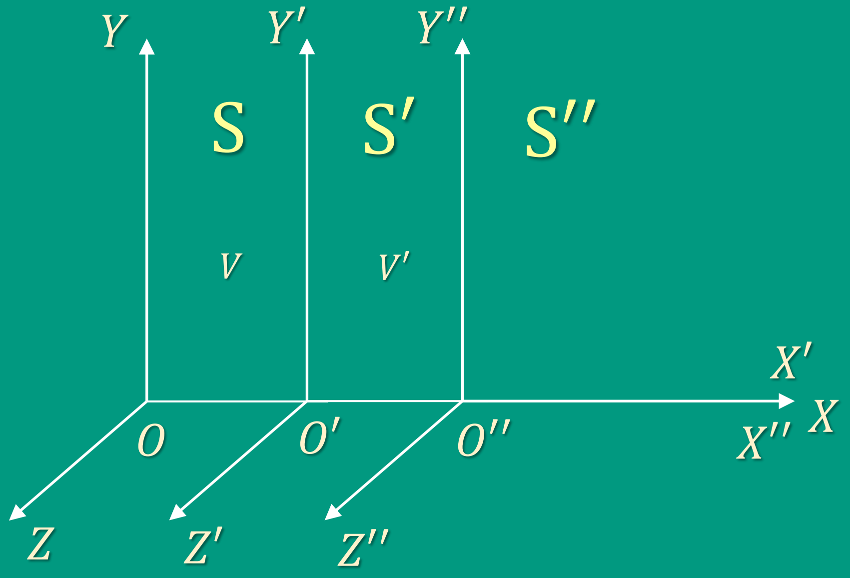
Let us consider a Lorentz transformation from S to S' system where S' moves away from S with velocity v. Then we have

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y, \quad z' = z, \\ t' &= \gamma\left(t - \frac{xv}{c^2}\right) \end{aligned}\right\} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (1)$$



Let us now consider a successeive Lorentz transformation from S' to S'' system where S'' moves away from S' with velocity v' relative to S'. Then we have

$$\left.\begin{aligned} x'' &= \gamma'(x' - v't') \\ y'' &= y', z'' = z' \\ t'' &= \gamma'\left(t' - \frac{x'v'}{c^2}\right) \end{aligned}\right\} \quad \gamma' = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}} \qquad (2)$$



Using (1) in (2) we get,

$$\begin{aligned} x'' &= \gamma'\left[\gamma(x - vt) - v'\gamma\left(t - \frac{xv}{c^2}\right)\right] \\ &= \gamma\gamma'\left[\left(1 + \frac{vv'}{c^2}\right)x - (v + v')t\right] \\ &= \gamma\gamma'\left(1 + \frac{vv'}{c^2}\right)\left[x - \left(\frac{v + v'}{1 + \frac{vv'}{c^2}}\right)t\right] \end{aligned} \qquad (3)$$

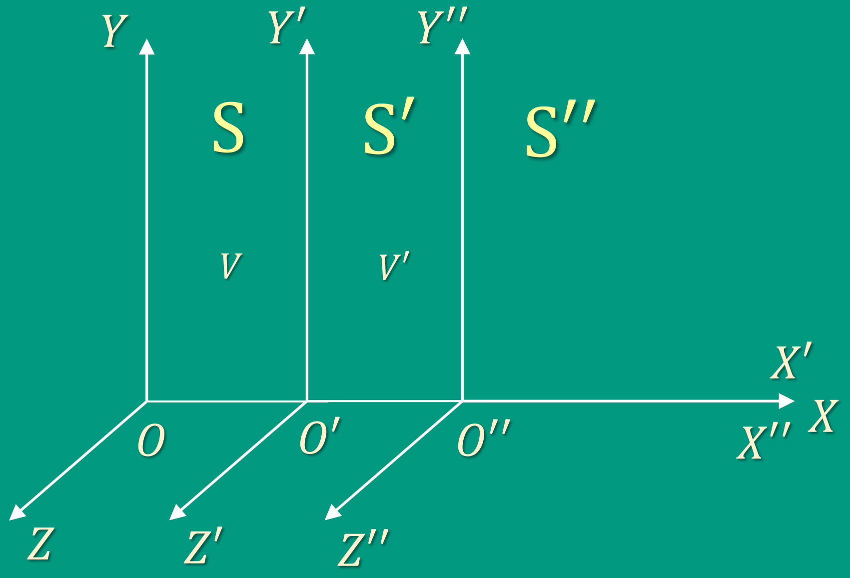
and

$$t'' = \gamma'\left(t' - \frac{x'v'}{c^2}\right)$$

$$= \gamma'\left[\gamma\left(t - \frac{xv}{c^2}\right) - \frac{v'}{c^2}\gamma(x - vt)\right]$$

$$= \gamma\gamma'\left[\left(1 + \frac{vv'}{c^2}\right)t - \left(\frac{v + v'}{c^2}\right)x\right]$$

$$= \gamma\gamma'\left(1 + \frac{vv'}{c^2}\right)\left[t - \frac{v + v'}{c^2}\left(\frac{1}{1 + \frac{vv'}{c^2}}\right)x\right] \quad (4)$$



If v'' be the resultant velocity of the velocities v and v' w.r.t. to S frame then by addition law of velocity,

$$v'' = \frac{v + v'}{1 + \frac{vv'}{c^2}}$$

Let us define

$$\gamma'' = \frac{1}{\sqrt{1 - \frac{v''^2}{c^2}}}$$

Now,

$$1 - \frac{v''^2}{c^2} = 1 - \frac{1}{c^2}\left(\frac{v + v'}{1 + \frac{vv'}{c^2}}\right)^2 = 1 - \frac{1}{c^2}\frac{(v + v')^2}{\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 + \frac{v^2 v'^2}{c^4} + \frac{2vv'}{c^2}\right) - (v^2 + v'^2 + 2vv')}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 + \frac{v^2 v'^2}{c^2} + 2vv' - v^2 - v'^2 + 2vv'}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 + \frac{v^2 v'^2}{c^2} - v^2 - v'^2}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2 - v^2 - v'^2 + \frac{v^2 v'^2}{c^2}}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 - \frac{v^2}{c^2}\right) - v'^2\left(1 - \frac{v^2}{c^2}\right)}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{c^2\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{\left(1 + \frac{vv'}{c^2}\right)^2}$$

$$\therefore \gamma'' = \frac{1}{\sqrt{1 - \frac{v''^2}{c^2}}} = \frac{1}{\sqrt{\frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}{\left(1 + \frac{vv'}{c^2}\right)^2}}}$$

$$= \frac{\left(1 + \frac{vv'}{c^2}\right)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{v'^2}{c^2}\right)}} = \gamma\gamma'\left(1 + \frac{vv'}{c^2}\right) \tag{5}$$

Using equation (5) in equations (3) and (4) we get,

$$x'' = \gamma''(x - v''t) \qquad t'' = \gamma''\left(t - \frac{xv''}{c^2}\right)$$

Also,

$$y'' = y', z'' = z'$$

Thus we see that we can get a direct Lorentz transformation from S to S'' system.

Putting, $v' = 0$ in equation (2) we get,

$$x'' = x', y'' = y', z'' = z', t'' = t'$$

which gives an identical Lorentz transformation from S'' to S' system. This is analogous to the *existence of identity* in a group.

Furthur setting $v' = -v$ we get $v'' = 0$. This shows that corresponding to every Lorentz transformation there *exists an inverse* Lorentz transformation.

∴ All the properties of a group are satisfied by Lorentz transformations and hence it is said to form a *group*.

Thank You