CALCULUS OF VARIATIONS

Calculus of variations, a branch of mathematics concerned with *the problem of finding a function* for which the value of a certain integral is either the largest or the smallest possible.

The problem of Calculus of variations was first solved by *Jacob Bernoulli* in 1696 but a general method of solving such problem was given by *Euler*.

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Recapitulate:

If $y = f(x)$ is a function, then necessary condition for y to be *maximum or minimum* is

$$\frac{dy}{dx} = 0$$

- If $\frac{d^2y}{dx^2} > 0$ then that point is minimum point.
- If $\frac{d^2y}{dx^2} < 0$ then that point is maximum point.

If $z = f(x, y)$ is a function, then necessary condition for z to be *maximum or minimum* is

$$\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial z}{\partial y} = 0$$

- If $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ then that point is minimum point.
- If $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$ and $\frac{\partial^2 z}{\partial y^2} < 0$ then that point is minimum point.

FUNCTIONS

$f: X \rightarrow Y$ $\qquad\qquad$ $y = f(x)$

FUNCTIONALS

$f: S \rightarrow R$

$S \rightarrow$ Set of functions $\qquad\qquad$ $R \rightarrow$ Real Numbers

$$S = \{f(x, y, y'): y = y(x), y' = y'(x)\}$$

$$I = \int_{x_1}^{x_2} f(x, y, y')dx$$

The calculus of variations is concerned with *maximum or minimum* of certain types of functions given in the form of integrals which are called *functionals* i.e. calculus of variation deals with the problem of finding a function $y(x)$ such that

$$I = \int_{x_1}^{x_2} f(x, y, y')dx$$

will be a maximum or minimum, also called an *extremum or stationary values*.

VARIATION OF A FUNCTION

Consider a function $f(x, y, y')$ where x is independent variable, $y = y(x)$ a dependent variable and $y' = y'(x) = \frac{dy}{dx}$.

Let y be changed from y to $y + h\alpha(x)$

and y' be changed from y' to $y' + h\alpha'(x)$

where h is a small parameter independent of x.

By *Taylor's expansion* theorem for two variables function we have

$$f\left(x, y + h\alpha(x), y' + h\alpha'(x)\right) = f(x, y, y') + \left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right) f +$$

$$+ \frac{1}{2!}\left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)^2 f + \ldots..$$

$$\therefore f\left(x, y + h\alpha(x), y' + h\alpha'(x)\right) - f(x, y, y') = h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'}$$

$$=> \delta f = h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'}$$ [Neglecting higher order terms]

Here, δf is called the variation of f.

Applying $f = y$, we get,

$$\delta y = h\alpha \frac{\partial y}{\partial y} + h\alpha' \frac{\partial y}{\partial y'} = h\alpha.1 + h\alpha'.0 \Rightarrow \delta y = h\alpha$$

Applying $f = y'$, we get,

$$\delta y' = h\alpha \frac{\partial y'}{\partial y} + h\alpha' \frac{\partial y'}{\partial y'} = h\alpha.0 + h\alpha'.1 \Rightarrow \delta y' = h\alpha'$$

$\therefore$ For $f(x, y, y')$

$$\delta f = \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'}$$

Thus geometrically, the variation δf represents the change in the value of f w. r. t. neighbouring curve considered.

Properties related to δ:

(1) δ and $\frac{d}{dx}$ are commutative i.e. $\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

(2) δ and $\int$ are commutative $\delta \int_{x_1}^{x_2} f(x, y, y') = \int_{x_1}^{x_2} \delta f(x, y, y')$

(3) If $f(x, y, y')$ and g(x, y, y') are functions then

(i) $\delta(c) = 0$ (ii) $\delta(c.f) = c.\delta f$ (iii) $\delta(f \pm g) = \delta(f) \pm \delta(g)$

(iv) $\delta(f.g) = f\delta(g) + g\delta(f)$ (v) $\delta(f/g) = \frac{g\delta(f) - f\delta(g)}{g^2}$

FUNCTIONAL

Let S be a set of functions of a single variable x defined over an interval (x_1, x_2). Then any function which assigns to each function in S a unique real number is called a functional i.e. functional is a mapping from functions to real numbers.

Now consider a function $f(x, y, y')$ and $x \in (x_1, x_2)$ then the integral

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')dx$$

is a functional.

For every $y(x)$, $I(y)$ gives a real number.

In calculus of variation, we determine the function $y = y(x)$, satisfying $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x, y, y')$, $I(y)$ is an *extremum*. A curve which satisfies this property is said to be an *extremal*.