GENERALIZED COORDINATES

The minimum number of independent coordinates or variables which is required to describe the motion of a dynamical system is known as *generalized coordinates*.

For a system of ***N*** particles and for ***k*** constraints and ***d*** dimension, the number of independent coordinates $(f) = dN - k$. These 'f' number of minimum independent coordinates required to describe configuration and motion of a mechanical system are called generalized coordinates and are denoted by $q_i (i = 1,2,3, \ldots, f)$.

Degree of Freedom: The minimum number of generalized coordinates required to completely describe the configuration of the system is called *degree of freedom*.

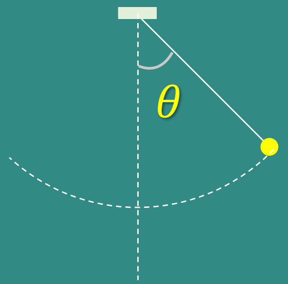
Generalized Coordinates	→	Minimum no. of Coordinates
Degree of freedom	→	No. of Generalized Coordinates

Generalized coordinates can be *any* set of parameters that equivalently specify a point in space.

We can express cartesian coordinates $\vec{r}_i$ in terms of generalized coordinates in the form

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, \dots, q_f, t)$$

SIMPLE PENDULUM



A *simple pendulum* consists of a point mass called bob suspended at the lower end of a massless and inextensible string of *constant length (l)* fixed at its upper end to a fixed rigid support.

Here, No. of free particles (N) = 1

No. of constraints (k) = 2

1st constraint = length of the string is constant

2nd constraint = the bob moves in a plane

Generalized Coordinates = 3N - k = 3.1 – 2 = 1

∴ Generalized Coordinate is given by θ.

Advantages of Generalized Coordinates:

- Generalized coordinates are *not limited* to Cartesian coordinates. They allow for the use of *alternative coordinate systems* that may be more suitable for describing the configuration of a specific system.
- Generalized coordinates provide a natural and convenient way *to handle constraints* in classical mechanics. By utilizing appropriate generalized coordinates, the constraints can be expressed as equations, simplifying the analysis and allowing for the incorporation of constraints directly into the *equations of motion*.
- Generalized coordinates enable a *more concise and elegant* representation of complex systems. By appropriately choosing the generalized coordinates, the degrees of freedom and independent variables necessary to describe the system can be significantly reduced.
- Generalized coordinates are closely tied to the concept of *energy and the Lagrangian formulation* in classical mechanics. The Lagrangian function, which is expressed in terms of generalized coordinates and their derivatives, simplifies the derivation of equations of motion using the principle of least action, providing a powerful and systematic approach *to solving problems* in classical mechanics.

- Generalized coordinates allow for system-specific descriptions that are tailored to the *unique properties* and *geometry* of the system under study.
- Many physical systems naturally possess non-Cartesian characteristics. Using generalized coordinates allows for a seamless *transition* between *different coordinate systems*, facilitating the analysis and understanding of systems with curved or non-rectangular geometries.

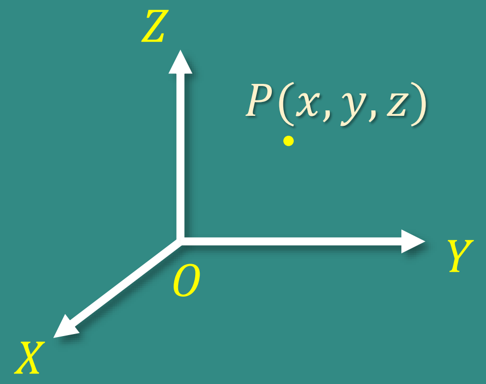
EULER'S THEOREM

If $f(\vec{r}) = f(x, y, z)$ then we have,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Or, $$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z$$

Let us consider *a free particle* in three dimensional coordinate system, then we have



No. of particle = 1
No. of constraint = 0
Degree of freedom (f) = 3N-k = 3.1 – 0 = 3
Generalized coordinates are (q_1, q_2, q_3).

For N no. of particles with k constraints the generalized coordinates are $(q_1, q_2, q_3, \ldots, q_f)$.

GENERALIZED DISPLACEMENT

Let us consider a *N-particle system* for which a small displacement $\delta\vec{r_i}$ is defined by change in position coordinates $\vec{r_i}(i = 1,2,,\dots,N)$ with time (t) kept as constant. The position vector $\vec{r_i}$ of the i^{th} particle in the form of generalized coordinates can be written as

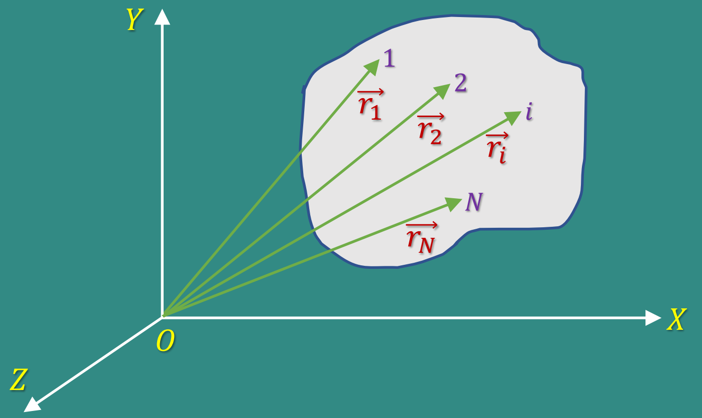
$$\vec{r_i} = \vec{r_i}(q_1, q_2, \dots, q_f, t)$$

Using Euler's theorem

$$\delta\vec{r_i} = \frac{\partial\vec{r_i}}{\partial q_1}\delta q_1 + \frac{\partial\vec{r_i}}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\vec{r_i}}{\partial q_f}\delta q_f + \frac{\partial\vec{r_i}}{\partial t}\delta t$$

$$=> \delta\vec{r_i} = \frac{\partial\vec{r_i}}{\partial q_1}\delta q_1 + \frac{\partial\vec{r_i}}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\vec{r_i}}{\partial q_f}\delta q_f$$

$$=> \delta\vec{r_i} = \sum_{j=1}^{f}\frac{\partial\vec{r_i}}{\partial q_j}\delta q_j$$



where δq_j represents *generalized displacement*.

GENERALIZED VELOCITY

Let us consider a dynamical system at *time t* comprised of *N particles*.

Let each particle be specified by the n generalized coordinates $q_1, q_2, q_3, \dots, q_n$. Then the time derivative of the generalized coordinates $q_j (j = 1,2,3, \dots, n)$ is called the *generalized velocity* which is denoted by $\dot{q}_j$.

The position vector $\vec{r_i}$ of the i^{th} particle in the form of generalized coordinates and time (t) can be written as

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \dots, q_f, t)$$

Using Euler's theorem

$$d\vec{r_i} = \frac{\partial \vec{r_i}}{\partial q_1} dq_1 + \frac{\partial \vec{r_i}}{\partial q_2} dq_2 + \cdots + \frac{\partial \vec{r_i}}{\partial q_f} dq_f + \frac{\partial \vec{r_i}}{\partial t} dt$$

$$=> \frac{d\vec{r_i}}{dt} = \frac{\partial \vec{r_i}}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \vec{r_i}}{\partial q_2} \frac{dq_2}{dt} + \cdots + \frac{\partial \vec{r_i}}{\partial q_f} \frac{dq_f}{dt} + \frac{\partial \vec{r_i}}{\partial t} \frac{dt}{dt}$$

$$=> \frac{d\vec{r_i}}{dt} = \frac{\partial\vec{r_i}}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\vec{r_i}}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial\vec{r_i}}{\partial q_f}\frac{dq_f}{dt} + \frac{\partial\vec{r_i}}{\partial t}$$

$$=> \vec{v} = \sum_{j=1}^{f} \frac{\partial\vec{r_i}}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial\vec{r_i}}{\partial t}$$

$$=> \vec{v} = \sum_{j=1}^{f} \frac{\partial\vec{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial\vec{r_i}}{\partial t}$$

where $\dot{q}_j$ is called generalized velocity.

GENERALIZED ACCELERATION

Let us consider a dynamical system at *time t* comprised of *N particles*.

Let each particle be specified by the n generalized coordinates $q_1, q_2, q_3, \dots, q_n$. Then the time derivative of the generalized velocity $\dot{q}_j (j = 1,2,3, \dots, n)$ is called the *generalized acceleration* which is denoted by $\ddot{q}_j$.

The velocity of the i^{th} particle in the form of generalized coordinates and time (t) can be written as

$$\vec{v} = \dot{\vec{r}_i} = \sum_{k=1}^{f} \frac{\partial \vec{r_i}}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r_i}}{\partial t} \qquad (1)$$

Differentiating both sides w.r.t. *'time'* we get

$$\dot{\vec{v}} = \ddot{\vec{r}_i} = \frac{d}{dt}\left[\sum_{k=1}^{f} \frac{\partial \vec{r_i}}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r_i}}{\partial t}\right]$$

$$\vec{a_i} = \frac{d}{dt}\left[\sum_{k=1}^{f}\frac{\partial \vec{r_i}}{\partial q_k}\dot{q}_k\right] + \frac{d}{dt}\left[\frac{\partial \vec{r_i}}{\partial t}\right]$$

$$= \sum_{k=1}^{f}\frac{\partial \dot{\vec{r_i}}}{\partial q_k}\dot{q}_k + \sum_{k=1}^{f}\frac{\partial \vec{r_i}}{\partial q_k}\ddot{q}_k + \frac{\partial \dot{\vec{r_i}}}{\partial t} \qquad (2)$$

Using equation (1) in equation (2) we get,

$$\vec{a_i} = \sum_{k=1}^{f}\frac{\partial}{\partial q_k}\left[\sum_{j=1}^{f}\frac{\partial \vec{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r_i}}{\partial t}\right]\dot{q}_k + \sum_{k=1}^{f}\frac{\partial \vec{r_i}}{\partial q_k}\ddot{q}_k + \frac{\partial}{\partial t}\left[\sum_{j=1}^{f}\frac{\partial \vec{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r_i}}{\partial t}\right]$$

$$=> \vec{a_i} = \sum_{k=1}^{f}\sum_{j=1}^{f}\frac{\partial^2 \vec{r_i}}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j + \sum_{k=1}^{f}\frac{\partial^2 \vec{r_i}}{\partial q_k \partial t}\dot{q}_k + \sum_{k=1}^{f}\frac{\partial \vec{r_i}}{\partial q_k}\ddot{q}_k + \sum_{j=1}^{f}\frac{\partial^2 \vec{r_i}}{\partial t \partial q_j}\dot{q}_j + \frac{\partial^2 \vec{r_i}}{\partial t^2}$$

GENERALIZED FORCE

Let us consider a *N-particle system* with no constraints imposed on the system.

Let a force $\sum_{i=1}^{N} F_i$ be acting on the system causing an arbitrary small displacement $\delta\vec{r_i}$ of the system by doing work δW which is given by

$$\delta W = \sum_{i=1}^{N} F_i . \delta\vec{r_i} \tag{1}$$

We have,

$$\delta\vec{r_i} = \sum_{j=1}^{3N} \frac{\partial\vec{r_i}}{\partial q_j} \delta q_j \tag{2}$$

Using (2) in (1) we get,

$$\delta W = \sum_{i=1}^{N} F_i . \sum_{j=1}^{3N} \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j$$

$$=> \delta W = \sum_{j=1}^{3N} Q_j . \delta q_j$$

$$Q_j = \sum_{i=1}^{N} F_i . \frac{\partial \vec{r_i}}{\partial q_j}$$

where Q_j is called *generalized force*.

GENERALIZED MOMENTUM

Let us consider a *N-particle system* with no constraints imposed on the system.

The K.E. T of the i^{th} particle having mass m_i and velocity v_i of a system is given by

$$T = \frac{1}{2} m_i v_i^2$$

$$=> T = \frac{1}{2} m_i \left(\frac{dx_i}{dt}\right)^2$$

$$=> T = \frac{1}{2} m_i \dot{x_i}^2 \qquad (1)$$

Differentiating equation (1) partially w.r.t. $\dot{x}_i$ we get,

$$\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2} m_i 2\dot{x}_i = m_i \dot{x}_i \qquad (2)$$

Linear momentum p_i of the particle is given by

$$p_i = m_i v_i$$

$$=> p_i = m_i \frac{dx_i}{dt}$$

$$=> p_i = m_i \dot{x}_i \qquad (3)$$

Comparing (2) and (3) we get,

$$p_i = \frac{\partial T}{\partial \dot{x}_i} \qquad (4)$$

Similarly, linear momentum is associated with generalized coordinate q_k called generalized momentum p_k is given by

$$p_k = \frac{\partial T}{\partial \dot{q}_k} \qquad (5)$$

First we derive the expression for K.E. (T) for a system of *N-particles* in terms of generalized velocities $\dot{q}_k$. K.E. of system of N-particles free from constraints is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i{}^2$$

$$v_i = \frac{dr_i}{dt} = \dot{r}_i$$

$$=> T = \sum_{i=1}^{N} \frac{1}{2} m_i (\vec{\dot{r}}_i . \vec{\dot{r}}_i) \qquad (6)$$

Now,

$$\vec{\dot{r}}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \qquad (7)$$

Using (7) and (6),

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \left[\sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right] . \left[\sum_{k=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N}\frac{1}{2}m_i\frac{\partial\vec{r_i}}{\partial q_j}\frac{\partial\vec{r_i}}{\partial q_k}\dot{q}_j\dot{q}_k + \sum_{i=1}^{N}\sum_{j=1}^{3N}\frac{1}{2}m_i\frac{\partial\vec{r_i}}{\partial q_j}\frac{\partial\vec{r_i}}{\partial t}\dot{q}_j +$$

$$+ \sum_{i=1}^{N}\sum_{k=1}^{3N}\frac{1}{2}m_i\frac{\partial\vec{r_i}}{\partial t}\frac{\partial\vec{r_i}}{\partial q_k}\dot{q}_k + \sum_{i=1}^{N}\frac{1}{2}m_i\left(\frac{\partial\vec{r_i}}{\partial t}\right)^2$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N}\frac{1}{2}m_i\frac{\partial\vec{r_i}}{\partial q_j}\frac{\partial\vec{r_i}}{\partial q_k}\dot{q}_j\dot{q}_k + \sum_{i=1}^{N}\sum_{k=1}^{3N}m_i\frac{\partial\vec{r_i}}{\partial q_k}\frac{\partial\vec{r_i}}{\partial t}\dot{q}_k + \sum_{i=1}^{N}\frac{1}{2}m_i\left(\frac{\partial\vec{r_i}}{\partial t}\right)^2$$

Differentiating partially w.r.t. $\dot{q}_k$

$$\frac{\partial T}{\partial\dot{q}_k} = \sum_{i=1}^{N}\sum_{j=1}^{3N}\frac{1}{2}m_i\frac{\partial\vec{r_i}}{\partial q_j}\frac{\partial\vec{r_i}}{\partial q_k}\dot{q}_j + \sum_{i=1}^{N}m_i\frac{\partial\vec{r_i}}{\partial q_k}\frac{\partial\vec{r_i}}{\partial t}$$

which is the generalized momentum.

GENERALIZED POTENTIAL

When the system is conservative: In this case the force acting on the system can be expressed as gradient of scalar potential function V i.e.

$$\vec{F} = -\nabla V \tag{1}$$

The work done by the force on the system during an arbitrary displacement $\delta\vec{r_i}$ of the system is

$$\delta W = \sum_{i=1}^{N} \vec{F_i} . \delta\vec{r_i} \tag{2}$$

Using (1) in (2) we get,

$$\delta W = \sum_{i=1}^{N} -\nabla_i V . \delta\vec{r_i}$$

$$= -\sum_{i=1}^{N} \left(\hat{\imath} \frac{\partial}{\partial x_i} + \hat{\jmath} \frac{\partial}{\partial y_i} + \hat{k} \frac{\partial}{\partial z_i} \right) V . \left(\delta x_i \hat{\imath} + \delta y_i \hat{\jmath} + \delta z_i \hat{k} \right)$$

$$= -\sum_{i=1}^{N} \left(\frac{\partial V}{\partial x_i} \hat{\imath} + \frac{\partial V}{\partial y_i} \hat{\jmath} + \frac{\partial V}{\partial z_i} \hat{k} \right) . \left(\delta x_i \hat{\imath} + \delta y_i \hat{\jmath} + \delta z_i \hat{k} \right)$$

$$= -\sum_{i=1}^{N} \left(\frac{\partial V}{\partial x_i} \delta x_i + \frac{\partial V}{\partial y_i} \delta y_i + \frac{\partial V}{\partial z_i} \delta z_i \right)$$

In terms of generalized coordinate q_k we have,

$$\delta W = -\sum_{k=1}^{3N}\left(\frac{\partial V}{\partial q_k}\delta q_k\right) \qquad => \delta W = \sum_{k=1}^{3N}\left(-\frac{\partial V}{\partial q_k}\right)\delta q_k$$

$$=> \delta W = \sum_{k=1}^{3N} Q_k \delta q_k$$

where $Q_k = -\frac{\partial V}{\partial q_k}$ represents the generalized force as workdone is the product of force and displacement. Here V is called *generalized potential* for a conservative system.

When the system is non-conservative: In this case the system depends on generalized velocities $\dot{q}_j$ besides q_j.

In this case the generalized force can be expressed as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

where $U = U(q_j, \dot{q}_j)$ is called velocity dependent *generalized potential* for a non-conservative system.